

6.207/14.15: Networks

Problem Set 4

Answer Keys

Problem 1

For some fixed $s_{-i} \in S_{-i}^0$, there exists $s_i^* \in S_i^0$ that maximizes $u_i(\cdot, s_{-i})$ by the Weierstrass extreme value theorem. Since s_i^* cannot be strictly dominated by any other strategy, s_i^* is in S_i^1 , that is, S_i^1 is nonempty. As u_i is continuous in s_i , S_i^1 is closed. Being a closed subset of a compact set, S_i^1 is compact. Thus, by induction we see that S_i^k is nonempty and compact for every k and i . Since S_i^k is a decreasing nested sequence of nonempty compact sets, we conclude that S_i^∞ is nonempty for every i by Cantor's intersection theorem.

Problem 2

Let $B_i : S_{-i} \rightrightarrows S_i$ be the best response correspondence for player i , that is, $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for every $s_i \in S_i$, $s_{-i} \in S_{-i}$, and $s_i^* \in B_i(s_{-i})$.

Fix $s_{-i} \in S_{-i}$. Since u_i is continuous in s_i and S_i is compact, $B_i(s_{-i})$ is nonempty and closed by the Weierstrass extreme value theorem. Since u_i is concave in s_i and S_i is convex, $B_i(s_{-i})$ is convex. Since u_i is continuous in (s_i, s_{-i}) , B_i is closed graph. Thus, by Kakutani's fixed-point theorem, there exists $s^* \in S$ such that $s_i^* \in B_i(s_{-i}^*)$ for every i . By definition, s^* is a pure strategy Nash equilibrium.

Problem 3

(a) Firm i maximizes the profit $(P - c_i)q_i = (a - q_i - q_j - c_i)q_i$. This induces the best response function $B_i(q_j) = (a - q_j - c_i)/2$. Solving the system of equations $B_1(q_2) = q_1$ and $B_2(q_1) = q_2$, we find the Nash equilibrium of this game $(q_1^*, q_2^*) = ((a + c_2 - 2c_1)/3, (a + c_1 - 2c_2)/3)$.

(b) Since $c_1 > c_2$, we have $q_1^* < q_2^*$.

(c) If we lower c_2 , not only firm 2 plumps up its equilibrium output but firm 1 cuts out its output in response to the strategic interaction between them. In total, the equilibrium aggregate output $Q^* = (2a - c_1 - c_2)/3$ will increase.

Problem 4

(a) Firm i earns profit $(p_i - 1/2)q_i$ where p_i is its own action and q_i is 4 if $p_i < p_j \wedge 4$, 2 if $p_i = p_j < 4$, and 0 otherwise. Thus, the normal form game is given by

	0	1	2	3	4+
0	(-1, -1)	(-2, 0)	(-2, 0)	(-2, 0)	(-2, 0)
1	(0, -2)	(1, 1)	(2, 0)	(2, 0)	(2, 0)
2	(0, -2)	(0, 2)	(3, 3)	(6, 0)	(6, 0)
3	(0, -2)	(0, 2)	(0, 6)	(5, 5)	(10, 0)
4+	(0, -2)	(0, 2)	(0, 6)	(0, 10)	(0, 0)

The pure-strategy Nash equilibria are $(p_1^*, p_2^*) = (1, 1)$ and $(2, 2)$.

(b) Now, the normal form game is given by

	0	1	2	3	4+
0	(-2, 0)	(-2, 0)	(-2, 0)	(-2, 0)	(-2, 0)
1	(0, -2)	(2, 0)	(2, 0)	(2, 0)	(2, 0)
2	(0, -2)	(0, 2)	(6, 0)	(6, 0)	(6, 0)
3	(0, -2)	(0, 2)	(0, 6)	(10, 0)	(10, 0)
4+	(0, -2)	(0, 2)	(0, 6)	(0, 10)	(0, 0)

Therefore, the only pure-strategy Nash equilibrium is $(p_1^*, p_2^*) = (1, 1)$.

(c) Compared to $(1, 1)$ in (a), firm 1 benefits for payoff 1 by its incumbency; however, relative to $(2, 2)$ in (a), firm 1 loses payoff of 1.

Problem 5

Define $\Phi : S \rightarrow \mathbb{R}$ by

$$\Phi(S) = |\{\text{red edges between same actions}\}| - |\{\text{blue edges between same actions}\}|.$$

Then, Φ is an exact potential.

Problem 6

- (a) Let x_1 , x_2 , and x_3 denote each flow of the upper, middle, and lower routes. Also, let a and b denote each flow that go to the middle route from the upper and lower nodes, respectively. In particular, $x_1 = 1 - a$, $x_2 = a + b$, and $x_3 = 1 - b$. The social cost is given by $(1 + x_1)x_1 + 3x_2^2 + (1 + x_3)x_3 = (2 - a)(1 - a) + 3(a + b)^2 + (2 - b)(1 - b)$. This can be minimized at $(a, b) = (3/14, 3/14)$, so the socially optimal routing is $(x_1^S, x_2^S, x_3^S) = (11/14, 3/7, 11/14)$ with the total cost of $47/14$.
- (b) The equilibrium routing will feature equal marginal costs across three paths, $1 + x_1 = 3x_2 = 1 + x_3$. Combining $x_1 + x_2 + x_3 = 2$, we obtain $(x_1^E, x_2^E, x_3^E) = (5/7, 4/7, 5/7)$. The total cost is $24/7$, yielding the welfare loss of $1/14$ from the optimum.
- (c) At the socially optimal routing, the marginal costs of the three routes are $25/14$, $9/7$, and $25/14$. To match these numbers, we need to impose a relative toll of $25/14 - 9/7 = 1/2$ to the middle route. We can balance the budget by imposing a reduced toll of $1/2 - c$ to the middle route and subsidizing c to the upper and lower routes, where c satisfies $(1/2 - c)x_2^S + c(x_1^S + x_3^S) = 0$; this gives $c = 9/40$.

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14.15J/6.207J Networks
Spring 2018

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