Massachusetts Institute of Technology
Departments of Electrical Engineering & Computer Science and Economics
6.207/14.15 Networks
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Problem Set 2

Problem 2.1
Let the directed graph G be a ring: node i is connected to $i + 1$ if $i < m$ and m is connected to 1. Compute both eigenvalue centrality and Katz centrality (with $\beta = \mathbf{1}$). Comment on your result. Do the same for a k-regular undirected network (i.e., an undirected network in which every vertex has degree k). You may find the steps outlined in Newman Problem 7.1 helpful. Comment on your result.

Problem 2.2
[Problem 7.2 from Newman] Suppose a directed network takes the form of a tree with all edges pointing inward towards a central vertex: (see figure in Newman). What is the PageRank centrality of the central vertex in terms of the single parameter α appearing in the definition of PageRank and the geodesic distances d_i from each vertex i to the central vertex?

Problem 2.3
Let the adjacency matrix A of a directed graph be nilpotent (i.e., $A^r = 0$ for some r).

(a) Give an example of such a graph with 5 nodes.

(b) Compute the eigenvalue centrality. Explain your answer.

(c) Compute the Katz centrality with $\beta = \mathbf{1}$. Explain your answer.

Problem 2.4
As flu season is upon us, we wish to have a Markov chain that models the spread of a flu virus. Assume a population of n individuals. At the beginning of each day, each individual is either infected or susceptible (capable of contracting the flu). Suppose that each pair (i, j), $i \neq j$, independently comes into contact with one another during the daytime with probability p. Whenever an infected individual comes into contact with a susceptible individual, he/she infects him/her. In addition, assume that overnight, any individual

who has been infected for at least 24 hours will recover with probability $0 < q < 1$ and return to being susceptible, independently of everything else (i.e., assume that a newly infected individual will spend at least one restless night battling the flu)

(a) Suppose that there are m infected individuals at daybreak. What is the distribution of the number of new infections by day end?

(b) Draw a Markov chain with as few states as possible to model the spread of the flu for $n = 2$. In epidemiology, this is called an SIS (Susceptible-Infected-Susceptible) model.

(c) Identify all recurrent states.

Due to the nature of the flu virus, individuals almost always develop immunity after contracting the virus. Consequently, we improve our model and assume that individuals become infected at most one time. Thus, we consider individuals as either infected, susceptible, or recovered.

(d) Draw a Markov chain to model the spread of the flu for $n = 2$. In epidemiology, this is called an SIR (Susceptible-Infected-Recovered) model.

(e) Identify all recurrent states.

Problem 2.5

There are n fish in a lake, some of which are green and the rest blue. Each day, Helen catches 1 fish. She is equally likely to catch any one of the n fish in the lake. She throws back all the fish, but paints each green fish blue before throwing it back in. Let G_i denote the event that there are i green fish left in the lake.

(a) Show how to model this fishing exercise as a Markov chain, where G_i are the states. Explain why your model satisfies the Markov property.

(b) Find the transition probabilities p_{ij}

(c) List the transient and recurrent states