Massachusetts Institute of Technology
Departments of Electrical Engineering & Computer Science and Economics
6.207/14.15 Networks
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## Problem Set 1

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**Problem 1.1**

[P7.3, Newman] Consider an undirected, connected tree of $n$ vertices. A particular edge in the tree joins vertices 1 and 2 and divides the tree into two disjoint regions of $n_1$ and $n_2$ vertices (also see Figure in Newman). Argue that the closeness centralities $C_1$ and $C_2$ of the two vertices, defined according to Equation (7.29) in Newman, are related by

$$\frac{1}{C_1} + \frac{n_1}{n} = \frac{1}{C_2} + \frac{n_2}{n}. \tag{1}$$

**Problem 1.2**

[P7.4, Newman] Consider an undirected, connected tree of $n$ vertices. Consider a particular vertex in the tree that has degree $k$. Naturally, its removal would divide the tree into $k$ disjoint trees with sizes being $n_1, \ldots, n_k$ where $n_1 + \cdots + n_k = n - 1$ and $n_1, \cdots, n_k \geq 1$.

(a) Show that the unnormalized betweenness centrality $x$ of the vertex, as defined in Eq. (7.36) in Newman, is

$$x = n^2 - \sum_{m=1}^{k} n_m^2. \tag{2}$$

Consider a special case of connected tree, a line graph of $n$ nodes: imagine placing $n$ nodes on a line next to each other and connecting only immediate neighbors, i.e. other than two end nodes all nodes connect to their neighbors of left and right, while end nodes connect to only one neighbor (also see Figure in Newman).

(b) Using (a) or otherwise, calculate the betweenness of the vertex that is $i$ hops away from one of the end vertex in the line graph for $i \geq 1$.

**Problem 1.3**
In many graphs, average path length and diameter are close to each other in value. But there are graphs in which they are very different.

(a) Describe an example of a graph where the diameter is more than three times as large as the average path length.

(b) Describe how you could extend your construction to produce graphs in which the diameter exceeds the average path length by as large a factor as you like (that is, for every number $c$, can you produce a graph in which the diameter is more than $c$ times as large as the average path length).

**Problem 1.4**
You are given an adjacency matrix of a network graph (p4-data.mat on the class website, under HW1). You can use the below python code to load the adjacency matrix. Using this, compute the following metrics of the associated graph:

(a) Clustering coefficient.

(b) Degree distribution. Plot the corresponding probability mass function.

(c) Average path length.

(d) Diameter.

**Problem 1.5**
Consider an undirected graph of $n = 2m$ nodes with the following properties:

1. nodes $1, \ldots m-1$ are fully connected,
2. nodes $m+1, \ldots 2m$ are fully connected,
3. nodes $m-1$, $m$, $m+1$ are connected,
4. there are no other edges.

Answer the following questions.

(a) Write some code to construct the adjacency matrix, say $A$, from some value $m$, for this graph. Explicitly compute and write down the adjacency matrix for $m = 3$.

(b) Let $D$ denote the diagonal matrix with $i$th diagonal entry being degree of node $i$. Compute the first and second eignvalues of matrix $L = AD^{-1}$ for $m = 5, 10, 15, 20$.

Consider a linear dynamical system over this graph with each node having a real value associated with it. Let $x_i(0)$ be the initial value associated with node $i$ at iteration 0, for $1 \leq i \leq 2m$. Let $x(0) = [x_i(0)]' \in \mathbb{R}^{2m}$ the vector of these initial values. The vector is updated iteratively as follows: at iteration $k+1, k \geq 0$,

$$x(k+1) = Lx(k). \tag{3}$$

(c) Assume that initial vector $x(0)$ is such that

$$x_i(0) = \begin{cases} 1, & 1 \leq i \leq m \\ 0, & m+1 \leq i \leq 2m. \end{cases} \tag{4}$$

For $m = 5, 10$, plot the values of $x_1(k), x_{m+1}(k)$ and $x_{2m}(k)$ for $k = 1, 10, 20, 50, 100$.

(d) Based on answers of (c), do you observe any peculiar behavior in $x(k)$? Can you explain it using the structure of $L$?

**Problem 1.6**

Romeo and Juliet are in *love*. Romeo positively reacts to Juliet; he loves her more if she shows him more love and he loves her less when she shows less. Juliet is a fickle lover; she loves Romeo more when he loves her less and vice versa. We want to model their love affair as a dynamical system in order to predict what will happen to them in the future. To do so, let $x(k)$ be the amount of love Romeo has for Juliet (measured in love units!), and let $y(k)$ be the amount of love Juliet has for Romeo. A simple dynamical system representing their interactions is given by: for some real numbers $a$ and $b$, the love at time $k+1$ is given by

$$x(k+1) = x(k) + ay(k) \qquad y(k+1) = bx(k) + y(k). \tag{5}$$

Assume that, initially $x(0), y(0) > 0$. Answer the following questions:

(a) Determine the signs of $a$ and $b$ to reflect the behavior of Romeo and Juliet.

(b) For what ranges of parameters $a$ and $b$ will Romeo's and Juliet's love fizzle away regardless of where they start?

(c) For what ranges of parameters $a$ and $b$ will Romeo and Juliet be forever caught in a cycle of love and hate?

(d) Both Romeo and Juliet were burnt before from loving someone else that does not love them. As a result, their love tomorrow discounts their own love today by a factor of 0.5. Rewrite the model and answer (a)-(b).

(e) What happens if both Romeo's and Juliet's love increases by one unit every single time regardless of the actions of the other? Answer questions (a)-(b).