Auctions 3:

Interdependent Values &

Linkage Principle

1 Interdependent (common) values.

- Each bidder receives private signal $X_i \in [0, w_i]$. ($w_i = \infty$ is possible)

- $(X_1, X_2, \ldots, X_n)$ are jointly distributed according to commonly known F ($f > 0$).

- $$V_i = v_i(X_1, X_2, \ldots, X_n).$$

  $$v_i(x_1, x_2, \ldots, x_n) \equiv E\left[V_i \mid X_j = x_j \text{ for all } j\right].$$

  Typically assumed that functional forms $\{v_i\}_{i=1}^N$ are commonly known.

- $v_i(0, 0, \ldots, 0) = 0$ and $E[V_i] < \infty$.

- Note the difference: The actual 'value' impact of information vs strategic effect of knowing the other's values.

- Symmetric case:

$$v_i(x_i, \mathbf{x}_{-i}) = v(x_i, \mathbf{x}_{-i}) = v(x_i, \pi(\mathbf{x}_{-i})).$$

- Affiliation: for all $\mathbf{x}', \mathbf{x}'' \in \mathcal{X}$,

$$f(\mathbf{x}' \vee \mathbf{x}'')f(\mathbf{x}' \wedge \mathbf{x}'') \geq f(\mathbf{x}')f(\mathbf{x}'').$$

  If f is twice continuously differentiable and strictly positive, affiliation is equivalent to

$$\frac{\partial^2 \ln f}{\partial x_i \partial x_j} \geq 0,$$

  also equivalent to require that $\ln f$ is a supermodular function (or that f is log-supermodular).

Examples:

- Art painting: experts, dealers, collectors. Some bidders are more informative than the others.

- Spectrum licenses: Each of the incumbents knows its own set of customers, interested in knowing characteristics of the rest.

- Oil field: each firm runs a test, obtains a noisy signal of capacity, ζ_i. The estimate of the expected value is $V_i = \frac{1}{N}\sum \zeta_i$, or, $V_i = a\zeta_i + (1 - a)\frac{1}{N-1}\sum_{j \neq i} \zeta_j$; (not all firms may run tests, ..., other components to value); $V_i = X_i + \frac{1}{N}\sum \zeta_i$.

- Job market? Employers bid for employees. Some employers can have better info.

2 Brief analysis

- Common values / Private values / Affiliated values / Interdependent values.
- Winner's curse.
- Second-price auction: Pivotal bidding—I bid what I get if I just marginally win.
- First-price auction: "Usual" analysis—differential equation, ...
- English auction: See below.
- Revenue ranking: *English* $>$ *SPA* $>$ *FPA*.

  (!) Interdependency and affiliation are important for the first part.

3 Second-price auction

Define

$$v(x,y) = E\left[V_1 \mid X_1 = x,\ Y_1 = y\right].$$

Equilibrium strategy

$$\beta^{\text{II}}(x) = v(x,x).$$

Indeed,

$$\begin{aligned} \Pi(b,x) &= \int_0^{\beta^{-1}(b)} \left(v(x,y) - \beta(y)\right) g(y|x)dy \\ &= \int_0^{\beta^{-1}(b)} \left(v(x,y) - v(y,y)\right) g(y|x)dy. \end{aligned}$$

Π is maximized by choosing $\beta^{-1}(b) = x$, that is, $b = \beta(x)$.

4 Example

1. Suppose S_1, S_2, and T are uniformly and independently distributed on $[0,1]$. There are two bidders, $X_i = S_i + T$. The object has a common value
$$V = \frac{1}{2}(X_1 + X_2).$$

2. In this example, in the first price auction:
$$\beta^{\mathsf{I}}(x) = \frac{2}{3}x, \qquad E[R^I] = \frac{7}{9}.$$

3. In the second-price auction $v(x,y) = \frac{1}{2}(x+y)$ and so
$$\beta^{\mathsf{II}}(x) = x, \qquad E[R^I] = \frac{5}{6}.$$

5 English auction (general case)

A strategy $\beta_i^{EA}(x_i)$ is a collection of strategies

$$\{\beta_i^N(x_i), \beta_i^{N-1}(x_i, z_N), \ldots, \beta_i^2(x_i, z_3, \ldots, z_N)\}.$$

Here z_k is the "revealed" type of the player who have exited when there were k players still active.

β_i^k is an intended exit price if no other player exited before.

How to construct? (Note that revealing need not to happen in equilibrium)

Remember, $\forall i$, $V_i(x_1, \ldots, x_n)$ is increasing in x_i (st) and in x_{-i} (wk), plus SC. (+other requirements)

Consider some p (suppose all bidders are active).

?Exists $z_i(p)$ for each i that with $x_i > z_i$, i is active, with $x_i < z_i$ is not.

Everyone can make these inferences. Plug into Vs.

For i, $V_i(x_i, x_{-i}) \geq V_i(x_i, z_{-i}) \geq V_i(z_i, z_{-i}) \lesseqqgtr p$.

In equilibrium, $V_i(z_i, z_{-i}) = p$.

Equilibrium: Solution to $\mathbf{V}(\mathbf{z}) = \mathbf{p}$. Once someone exits, fix his z, continue.

6 Linkage principle

Define

$$W^A(z, x) = E\left[P(z) \mid X_1 = x,\ Y_1 < z\right]$$

expected price paid by the winning bidder when she receive signal x but bids z.

Proposition: (Linkage principle):

Let A and B be two auction forms in which the highest bidder wins and (she only) pays positive amount. Suppose that symmetric and increasing equilibrium exists in both forms. Suppose also that

1. for all x, $W_2^A(x, x) \geq W_2^B(x, x)$.
2. $W^A(0, 0) = W^B(0, 0) = 0$.

Then, the expected revenue in A is at least as large as the expected revenue in B.

So,the greater the linkage between a bidder's own information and how he perceives the others will bid the greater is the expected price paid upon winning.

Auctions 4:

Multiunit Auctions &

Cremer-McLean Mechanism

M units of the same object are offered for sale.

Each bidder has a set of (marginal values) $V^i = (V_1^i, V_2^i, \ldots V_M^i)$, the objects are substitutes, $V_k^i \geq V_{k+1}^i$.

Extreme cases: unit-demand, the same value for all objects.

- Types of auctions:
- The discriminatory ("pay-your-bid");
- Uniform-price;
- Vickrey;
- Multi-unit English;

- Ausubel;

- Dutch, descending uniform-price,

- ...

Issues: Existence and description of equilibria, price series if sequential, efficiency, optimality, non-homogenous goods, complementarities,...

7 Vickrey Auction

- Let $(b_1^i, b_2^i, \ldots, b_n^i)$ be the vector of bids submitted by i.

- Winners: M highest bids.

- Payments: If player i wins m objects, then has to pay the sum of m highest non-winning bids from the others.

  Or, price for each unit is: minimal value to have and win.

  E.g. to win 3d unit need to bid among $(M-2)$ highest bids, $p = (M-2)$sd highest bid of the others.

- Weakly dominant to bid truthfully, $b_k^i = V_k^i$.

8 Interdependent valuations

8.1 Notation

K objects; given $\mathbf{k} = (k_1, \ldots, k_N)$, denote

$$\mathbf{V}^{\mathbf{k}} = \left(V_1^{k_1}, \ldots, V_N^{k_N}\right).$$

Winners circle at s, $\mathcal{I}^{\mathbf{k}}(\mathrm{s})$, is the set of bidders with the highest value among $\mathbf{V}^{\mathbf{k}}$.

$\mathbf{k}$ is *admissible* if $1 \leq k_i \leq K$ and

$$0 \leq \sum_{i=1}^{N} (k_i - 1) < K.$$

8.2 Single-crossing condition

MSC (*single-crossing*) For any admissible $\mathbf{k}$, for all $\mathbf{x}$ and any pair of players $\{i, j\} \subset \mathcal{I}^{\mathbf{k}}(\mathbf{x})$,

$$\frac{\partial V_i^{k_i}(\mathbf{x})}{\partial x_i} > \frac{\partial V_j^{k_j}(\mathbf{x})}{\partial x_i}.$$

8.3 Efficiency: VCG mechanism (generalized Vickrey auction)

- Allocation rule: Efficient.

- Payments: Vickrey price that player j pays for kth unit won:

$$p_j^k = V_j^k(s_j^k, x_{-j}) =$$

$(M-k+1)$th highest

among $\{V_i^m(s_j^k, x_{-j})\}_{i\neq j}^{m=1..M}$.

These are generically different across units and winners (unlike with private values).

9 Cremer & McLean Mechanism

- Multiple units. Single-crossing and non-independent values.

- Efficient, Extract all the surplus.

Discrete support: $\mathcal{X}^i = \{0, \Delta, 2\Delta, \ldots, (t_i - 1)\Delta\}$, discrete single-crossing is assumed (no need if the values are private).

$\Pi(\mathbf{x})$ is the joint probability of x, $\Pi_i = (\pi(\mathbf{x}_{-i}|x_i))$.

Theorem: In the above conditions and if Π has a full rank, there exists a mechanism in which truth-telling is an efficient ex post equilibrium and in which the seller extracts full surplus from the bidders.

Proof: Consider VCG mechanism $(\mathbf{Q}^*, \mathbf{M}^*)$. Define,

$$U_i^*(x_i) = \sum_{\mathbf{x}_{-i}} \pi(\mathbf{x}_{-i}|x_i)\left[Q_i^*(\mathbf{x})V_i(\mathbf{x}) - M_i^*(\mathbf{x})\right].$$

This is the expected surplus of buyer i in VCG mechanism. Define, $\mathbf{u}_i^* = (U_i^*(x_i))_{x_i \in \mathcal{X}^i}$.

There exists $\mathbf{c}_i = (c_i(\mathbf{x}_{-i}))_{\mathbf{x}_{-i} \in \mathcal{X}_{-i}}$, such that $\Pi_i \mathbf{c}_i = \mathbf{u}_i^*$. Equivalently,

$$\sum_{\mathbf{x}_{-i}} \pi(\mathbf{x}_{-i}|x_i) c_i(\mathbf{x}_{-i}) = U_i^*(x_i).$$

Then, *CM* mechanism $(\mathbf{Q}^*, \mathbf{M}^{CM})$ is defined by

$$M_i^{\mathbf{CM}}(\mathbf{x}) = M_i^*(\mathbf{x}) + c_i(\mathbf{x}_{-i}).$$

Remarks:

- Private values (correlated), equiv. second price auction with additional payments.

- Negative payoffs sometimes, not ex post IR, payoffs arbitrarily large if the distribution converges to the independent one.