14.123 Microeconomic Theory III
Final Exam
March 18, 2010
(80 Minutes)

1. (30 points) This question assesses your understanding of expected utility theory.

(a) In the following pair of games, check whether the players' preferences over lotteries on the strategy profiles are identical (i.e. row player's preferences on the left to the row player's preferences on the right and column player's preferences on the left to the column player's preferences on the right).

	L	M	R
a	2,-2	1,1	-3,7
b	1,10	0,4	0,4
c	-2,1	1,7	-1,-5

	L	M	R
a	12,-1	5,0	-3,2
b	5,3	3,1	3,1
c	-1,0	5,2	1,-2

Answer: For the preferences to be the same, it must be that $v_i = a_i u_i + b_i$ for some a_i and b_i where a_i is positive and u_i and v_i are the utility functions of i in the left and the right game, respectively. Since $u_1(b, M) = 0$ and $v_1(b, M) = 3$, we must have $b_1 = 3$ (Row player is 1 and the column player is 2.) Since $u_1(c, M) = 1$ and $v_1(c, M) = 5$, we then have $a_1 = 2$. But $v_1(a, L) = 12 \neq 2 \cdot 2 + 3 = 2u_1(a, L) + 3$. Therefore, Player 1's preferences are different in two games. On the other hand, one can easily see that $u_2 = 3v_2 + 1$, showing that the preferences of Player 2 are identical in the two games.

(b) Under Postulates P1-5 of Savage, let $D_1, D_2, \ldots, D_n$ be disjoint non-null events such that $D_1 \dot{\sim} D_2 \dot{\sim} \cdots \dot{\sim} D_n$, where $\dot{\succeq}$ and $\dot{\sim}$ are the at least as likely as and as likely as relations between events, derived from betting preferences as in the class. Given any subsets N and N' of $\{1, 2, \ldots, n\}$, show that

$$\bigcup_{i \in N} D_i \dot{\succeq} \bigcup_{i \in N'} D_i \iff |N| \geq |N'|.$$

Answer: Under P1-5, $\dot{\succeq}$ is a qualitative probability, as you have shown in your homework. In particular, for any disjoint events B, C, D where C is not null,

$$C \ \dot{\succ} \ \varnothing \tag{1}$$
$$B \cup C \ \dot{\succeq} \ B \cup D \iff C \dot{\succeq} D. \tag{2}$$

I will show that if $|N| = |N''|$, then $\bigcup_{i \in N} D_i \dot{\sim} \bigcup_{i \in N''} D_i$. This also implies that if $|N'| > |N|$, then $\bigcup_{i \in N'} D_i \dot{\succ} \bigcup_{i \in N} D_i$. [Proof: Take N'' to be a subset of N' with $|N| = |N''|$. Take also $B = \bigcup_{i \in N''} D_i$, $C = \bigcup_{i \in N' \setminus N''} D_i$, which is not null, and $D = \varnothing$. By (2),

$$\bigcup_{i \in N'} D_i = B \cup C \dot{\succ} B \cup D = B \dot{\sim} \bigcup_{i \in N} D_i,$$

where the last indifference is by the fact I am about to show.] To show this fact, use mathematical induction on $|N|$. If $|N| = |N''| = 0$, both sides are empty sets, and indifference is true. Suppose that the indifference holds for $|N| = |N''| = m$, and consider N, N'' with $|N| = |N''| = m + 1$. Let $M = N \backslash \{i^*\}$ and $M'' = N'' \backslash \{j^*\}$ for some $i^* \in N$, $j^* \in N''$. Clearly, $|M| = |M''| = m$, and by inductive hypothesis,

$$\bigcup_{i \in M} D_i \dot{\sim} \bigcup_{i \in M''} D_i. \tag{3}$$

Then,

$$\bigcup_{i \in N} D_i = \left(\bigcup_{i \in M} D_i \right) \cup D_{i^*} \dot{\sim} \left(\bigcup_{i \in M} D_i \right) \cup D_{j^*} \dot{\sim} \left(\bigcup_{i \in M''} D_i \right) \cup D_{j^*} = \bigcup_{i \in N''} D_i,$$

where the first indifference is by (2) and the assumption that $D_{i^*} \dot{\sim} D_{j^*}$, and the second indifference is by (2) and (3).

2. (30 points) Consider an expected profit maximizing monopolist who faces an uncertain demand. He supplies q units of goods at zero cost and sells it at price $\theta - q$, where θ is unknown. [The price and the supply level can be negative.]

   (a) Assuming that $\theta \sim N(y, \sigma^2)$, compute the monopolist's optimal supply q and his expected profit under the optimal supply.

   Answer: The expected payoff of the monopolist is

   $$E[u] = q(E[\theta] - q) = q(y - q).$$

   Hence, the optimal q is

   $$q^* = y/2.$$

   Thus, the payoff under optimal q^* is

   $$U(y) = y^2/4.$$

   (b) Suppose that, through market research, the monopolist can learn about θ. In particular, by investing c^2, he can learn the value of a random variable Y before choosing his supply q, such that $\theta = X + Y$, $X \sim N(0, 1 - c)$ and $Y \sim N(0, c)$. How much should the monopolist invest? [Note that the utility function of the monopolist is $(\theta - q)q - c^2$.]

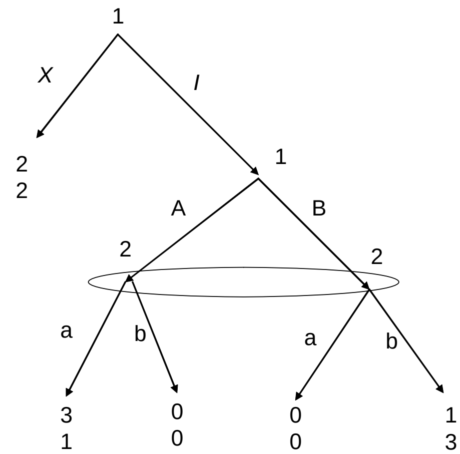
   Answer: Conditional on Y, θ is distributed with $N(Y, 1 - c)$. Hence, by part (a), conditional on Y, the expected payoff of the monopolist under optimal supply is $Y^2/4$. Therefore, the expected utility from c is

   $$V(c) = E\left[Y^2/4\right] - c^2 = E\left[Y^2\right]/4 - c^2 = c/4 - c^2,$$

   where the first equality is by the law of iterated expectations and the last equality is by the fact that $Y \sim N(0, c)$. Therefore, the optimal c is

   $$c^* = 1/8.$$

3. (40 points) Consider the reduced normal form of the following game, in which the strategy set of Player 1 is $\{X, A, B\}$, so that the equivalent strategies XA and XB are represented by a single strategy X.



(a) Compute the set of rationalizable strategies. (Show your result.)

Answer: The reduced game in normal form is

	a	b
X	2,2	2,2
A	3,1	0,0
B	0,0	1,3

Note that X strictly dominates B. Hence, B is eliminated. The remaining game is

	a	b
X	2,2	2,2
A	3,1	0,0

Nothing is eliminated in the new game (X is a best reply to b; A is a best reply to a; a is a best reply to A and b is a best reply to X). Therefore, $S^\infty = \{X, A\} \times \{a, b\}$.

(b) Compute the set correlated equilibria. (Show your result.)

Answer: Recall that a correlated equilibrium would put 0 probability on non-rationalizable strategy profiles. Hence, a correlated equilibrium is $p = (p_1, p_2, p_3, p_4)$ with $p_1 + p_2 + p_3 + p_4 = 1$ where the probabilities are as in the table

	a	b
X	p_1	p_2
A	p_3	p_4

Note that it must be that $p_4 = 0$ because if $p_4 > 0$, Player 2 would strictly prefer to play a when she is asked to play b. We thus have

	a	b
X	p_1	p_2
A	p_3	0

Since a weakly dominates b (as B has zero probability), she does not deviate when she is asked to play a. Likewise, when Player 1 is asked to play A, he knows that she plays a, and he does not deviate. When player 1 is asked to play X, he assigns probability $p_1/(p_1+p_2)$ on a and $p_2/(p_1+p_2)$ on b. Hence, the payoff from X is 2 and the payoff from A is $3p_1/(p_1+p_2)$. Thus, he does not deviate iff

$$p_1 \leq 2p_2.$$

Therefore, the set of correlated equilibria is

$$\{p|p(X,a) \leq 2p(X,a), p(A,b) = p(B,a) = p(B,b) = 0\}.$$

(c) Suppose that in addition to the type with the payoff function above, with probability 0.1, Player 1 has a "crazy" type who gets 1 if he plays A and 0 otherwise. Compute the set of all sequential equilibria.

Answer: The only sequentially rational plan for the crazy type is IA. Hence, the information set of Player 2 is reached. Moreover, for the normal type of player 1, IB is not sequentially rational under any belief by part (a). Hence, at her information set, Player 2 assigns probability 1 on the event that player 1 will play A in the subgame. [Proof: she can assign positive probability only on the strategies in $\{XA, XB, IA\}$ for the normal type. Conditional on I, either she assigns zero probability on normal type, concluding that she faces the crazy type and he will play A with probability 1, or she assigns probability 1 on that the normal type plays A, in which case both types play A.] Therefore, by sequential rationality, in any sequential equilibrium, she must play a with probability 1. Of course, this implies that the normal type plays IA with probability 1 in any sequential equilibrium. Therefore, the unique sequential equilibrium is

$$\begin{aligned} s_1(normal) &= IA \\ s_1(crazy) &= IA \\ s_2 &= A \\ \Pr(normal|I) &= 0.9. \end{aligned}$$