Problem 1

Aside: Many of you did not rigorously verify that the assessment you found was consistent. In particular, I mentioned in recitation that if σ is already completely mixed then there is no need to construct a sequence satisfying the conditions in the definition of consistency, but in this problem σ^* is *not* completely mixed, so you have to construct such a sequence.

Call the left node in player 2's information set n_1 and right node (the one following I) n_2.

Then (σ^*, μ^*) is a sequential equilibrium where $\sigma^* = \left(\frac{1}{9}I + \frac{8}{9}X, b; \frac{1}{3}L + \frac{2}{3}R\right)$, $\mu^*(n_1) = \frac{3}{4}$, and $\mu^*(n_2) = \frac{1}{4}$ (i.e., player 1 plays I with probability $\frac{1}{9}$ and plays b, and player 2 plays L with probability $\frac{1}{3}$). To check that (σ^*, μ^*) is a sequentially equilibrium, I check sequential rationality and consistency:

Sequential rationality: Player 1's expected payoff from X is 2, and her expected payoff from I is $\left(\frac{1}{3}\right)0 + \left(\frac{2}{3}\right)3 = 2$, so her play is sequentially rational at her first node. Her play is clearly sequentially rational at her second node. Player 2's expected payoff from L is $\left(\frac{3}{4}\right)0 + \left(\frac{1}{4}\right)4 = 1$, and his expected payoff from R is 1, so his play is sequentially rational.

Consistency: Let $\sigma^m \equiv \left(\frac{1}{9}I + \frac{8}{9}X, \left(\frac{m-1}{m}\right)b + \left(\frac{1}{m}\right)a; \frac{1}{3}L + \frac{2}{3}R\right)$, $\mu^m(n_1) \equiv \frac{3}{4}$, and $\mu^m(n_2) \equiv \frac{1}{4}$. For all m, σ^m is completely mixed, $\mu^m(n_1) = \frac{\sigma^m(n_1)}{\sigma^m(n_1)+\sigma^m(n_2)} = \frac{1/4}{(1/4)+(3/4)(1/9)} = \frac{3}{4}$, and $\mu^m(n_2) = \frac{\sigma^m(n_2)}{\sigma^m(n_1)+\sigma^m(n_2)} = \frac{1}{4}$, so this sequence satisfies the conditions in the definition of consistency.

Problem 2

(a) Let μ_{2t} be player 2's assessment of the probability that player 1 is irrational at his time-$2t$ information set, let p_t be the probability that (rational) player 1 plays *exit* at her time-t node, and let q_t be the probability that player 2 plays *exit* at his time-t node. These parameters define an assessment, with the understanding that the players mix independently across nodes. Let t^* be the largest integer $t \in \{1, 2, \ldots T\}$ such that $\left(\frac{1}{3}\right)^{T-t} < \varepsilon$ if such an integer exists, and otherwise let $t^* = 0$. A sequential equilibrium is given by:

$$
\begin{aligned}
\mu_{2t} &= \left(\frac{1}{3}\right)^{T-t} \text{ for } t = t^*+1, t^*+2, \ldots, T \\
\mu_{2t} &= \varepsilon \text{ for } t = 1, 2, \ldots, t^* \\
p_t &= \frac{2/3}{1-\mu_{t-1}} \text{ for } t = 2t^*+3, 2t^*+5, \ldots, 2T-1 \\
p_{2t^*+1} &= \frac{1-3^{T-t^*-1}\varepsilon}{1-\varepsilon} \\
p_t &= 0 \text{ for } t = 1, 3, \ldots, 2t^*-1 \\
q_{2T} &= 1 \\
q_t &= \frac{2}{3} \text{ for } t = 2t^*+2, 2t^*+4, \ldots, 2T-2 \\
q_t &= 0 \text{ for } t = 2, 4, \ldots, 2t^*.
\end{aligned}
$$

To check this, check sequential rationality and consistency.

Sequential rationality for player 1: At $2T-1$, *exit* is sequentially rational and indeed $p_{2T-1} = 1$. At $t \in \{2t^*+1, 2t^*+3, \ldots, 2T-1\}$, *exit* yields payoff $t+1$ and *stay* yields payoff

$$(q_{t+1})\, t + (1-q_{t+1})\,(t+2) = \left(\frac{2}{3}\right) t + \left(\frac{1}{3}\right)(t+3) = t+1,$$

so any mix of *exit* and *stay* is sequentially rational. At $t \in \{1, 3, \ldots, 2t^*-1\}$, *exit* yields payoff $t+1$ and *stay* yields payoff at least $t+3$ (because 2 plays *stay* at $t+1$ with probability 1, and 1 has the option of playing *exit* at $t+2$ and therefore must receive expected payoff of at least $t+3$ conditional on reaching node $t+2$ in any sequential equilibrium). So at $t \in \{1, 3, \ldots, 2t^*-1\}$ *stay* is sequentially rational, and indeed $p_t = 0$.

Sequential rationality for player 2: At $2T$, *exit* is sequentially rational and indeed $q_{2T} = 1$. At $t \in \{2t^*+2, 2t^*+4, \ldots, 2T-2\}$, *exit* yields payoff $t+1$ and *stay* yields payoff

$$
\begin{aligned}
&((1-\mu_t)\, p_{t+1})\, t + (1-(1-\mu_t)\, p_{t+1})\,(t+3) \\
= &\left((1-\mu_t)\frac{2/3}{1-\mu_t}\right) t + \left(1-(1-\mu_t)\frac{2/3}{1-\mu_t}\right)(t+3) = t+1,
\end{aligned}
$$

so any mix of *exit* and *stay* is sequentially rational. At $t \in \{2, 4, \ldots, 2t^*-2\}$, *exit* yields payoff $t+1$ and *stay* yields payoff at least $t+3$ (by the same argument as for player 1), so *stay* is sequentially rational and indeed $q_t = 0$. Finally, at $t = 2t^*$, *exit* yields payoff $2t^*+1$

and *stay* yields payoff

$$
\begin{aligned}
& \left(\left(1-\varepsilon\right) p_{2t^*+1}\right)\left(2t^*\right)+\left(1-\left(1-\varepsilon\right) p_{2t^*+1}\right)\left(2t^*+3\right) \\
= \; & \left(1-3^{T-t^*-1}\varepsilon\right)\left(2t^*\right)+\left(3^{T-t^*-1}\varepsilon\right)\left(2t^*+3\right).
\end{aligned}
$$

By definition of t^*, $3^{T-t^*-1}\varepsilon > \frac{1}{3}$, so this payoff is greater than $2t^*+1$. Therefore, *stay* is sequentially rational, and indeed $q_t = 0$.

Consistency: Let $\tilde{\mu}_{2t}$ be the conditional probability that player 1 is irrational at time $2t$ under the above strategy profile; we wish to show that $\tilde{\mu}_{2t} = \mu_{2t}$ for all $t \in \{1, 2, \ldots, T\}$. If $t \in \{1, 2, \ldots, t^*\}$, then clearly $\tilde{\mu}_{2t} = \varepsilon$. By Bayes' rule,

$$
\begin{aligned}
\tilde{\mu}_{2(t^*+1)} &= \frac{\tilde{\mu}_{2t^*}}{1-\left(1-\tilde{\mu}_{2t^*}\right) p_{2t^*+1}} \\
&= \frac{\varepsilon}{1-\left(1-\varepsilon\right)\left(\frac{1-3^{T-t^*-1}\varepsilon}{1-\varepsilon}\right)} \\
&= \frac{1}{3^{T-(t^*+1)}}
\end{aligned}
$$

If $t \in \{t^*+2, t^*+4, \ldots, T\}$, then by Bayes' rule

$$
\tilde{\mu}_{2t} = \frac{\tilde{\mu}_{2(t-1)}}{1-\left(1-\mu_{2(t-1)}\right) p_{2t-1}} = \frac{\tilde{\mu}_{2(t-1)}}{1/3} = 3\tilde{\mu}_{2(t-1)},
$$

and therefore $\tilde{\mu}_{2t} = \left(\frac{1}{3}\right)^{T-t}$.

Since the above strategy profile is not completely mixed, technically one should construct a sequence of completely mixed strategy profiles satisfying the conditions in the definition of consistency. This can be done by simply assuming that each player plays *exit* with probability $\frac{1}{m}$ in the m^{th} strategy profile in the sequence, as in Problem 1.

(b) Player 1 stays with probability 1 at $t = 1$ if and only if either $t^* \geq 1$ or $t^* = 0$ and $p_{2t^*+1} = p_1 = 1$. This holds if and only if $\left(\frac{1}{3}\right)^{T-1} \leq \varepsilon$.

Note that $\left(\frac{1}{3}\right)^{T-1}$ decreases exponentially in T. Therefore, ε must be extremely small for player 1 to exit with positive probability at $t = 1$. Recalling that player 1 exits with probability 1 at $t = 1$ in the complete information game, this shows that even a very small probability of irrationality has a large effect on player 1's sequential equilibrium behavior.

(c) The statement is false, by part (a) and the fact that every sequential equilibrium is a Nash equilibrium. However, note that $t^* = 0$ for sufficiently small ε, and in this case

$p_1 = \frac{1-3^{T-1}\varepsilon}{1-\varepsilon}$, which converges to 1 as $\varepsilon \to 0$. Therefore, there is continuity at $\varepsilon = 0$ in the sense that player 1 exits at $t = 1$ with probability close to 1 when ε is sufficiently small.

Problem 3

There is an important subtlety here. First, note that as stated the statement in the problem is false, and problem 2.a is a counterexample: In problem 2.a, for any small $\varepsilon > 0$, player 2 plays *exit* with probability $\frac{2}{3}$ at $2T-2$ in sequential equilibrium, but if $\varepsilon = 0$ then player 2 plays *exit* with probability 1 at $2T-2$. This shows that the set of sequential equilibrium may fail to be upper hemi-continuous in beliefs if nature assigns probability 0 to some actions. See pages 341-342 of Fudenberg and Tirole for an informative discussion of this point.

There are two things we can do to restore upper hemi-continuity and thus make the statement in the problem true. The first is to assume that nature assigns positive probability to every action. The second is to modify the definition of sequential equilibrium by assuming that nature trembles in the same way the players do; formally, this means specifying that consistency of (σ, μ) in game G means that there is a sequence (σ^m, p^m, μ^m) converging to (σ, p, μ), where p is the distribution over nature's moves in game G, such that σ^m **and** p^m are completely mixed and μ^m is deriving using Bayes' rule from σ^m and p^m.

I take the second approach here. This is exactly equivalent to identifying the games G^m and G^* and adding a player $n+1$ (nature) who follows strategy $\sigma^m_{n+1} = p^m$ in strategy profile σ^m, where p^m is the distribution of nature's moves in game G^m. In particular, the condition that $\sigma^m \to \sigma^*$ now implies that $p^m = \sigma^m_{n+1} \to \sigma^*_{n+1} = p^*$ (the distribution of nature's moves in game G^*), which is exactly the condition that $G^m \to G^*$ in the original formulation. I now check sequential rationality and consistency of (σ^*, μ^*):

Sequential rationality: That (σ^m, μ^m) is sequentially rational means that, for all $i \in N$, $s_i \in S_i$, and $h \in H_i$,

$$\sum_{x \in h} u_i \left(\sigma^m_i (h), \sigma^m_{-i} (h)\right) \mu^m (x|h) \geq \sum_{x \in h} u_i \left(s_i, \sigma^m_{-i} (h)\right) \mu^m (x|h).$$

Since u_i is continuous (because G^* is finite) and limits preserve weak inequalities, this implies that, for all $i \in N$, $s_i \in S_i$, and $h \in H_i$,

$$\sum_{x \in h} u_i \left(\sigma_i^* (h), \sigma_{-i}^* (h)\right) \mu^* (x|h) \geq \sum_{x \in h} u_i \left(s_i, \sigma_{-i}^* (h)\right) \mu^* (x|h).$$

This is precisely sequential rationality of (σ^*, μ^*).

Consistency: For all $m \in \mathbb{N}$, let $(\sigma^{m,n}, \mu^{m,n})_{n \in \mathbb{N}}$ be a sequence of assessments converging to (σ^m, μ^m) that satisfies the conditions in the definition of consistency (of (σ^m, μ^m)); in particular, $\sigma^{m,n}$ is completely mixed and $\mu^{m,n}(x|h) = \frac{\sigma^{m,n}(x)}{\sigma^{m,n}(h)}$ for all information sets h and all nodes $x \in h$. Define the distance between strategy profiles σ and σ' by

$$d(\sigma, \sigma') \equiv \max_{i \in N, s_i \in S_i} |\sigma(s_i) - \sigma'(s_i)|,$$

and define the distance between probability distributions μ and μ' by

$$\tilde{d}(\mu, \mu') \equiv \max_{h, x \in h} |\mu(x|h) - \mu'(x|h)|.$$

Choose $n_m \in \mathbb{N}$ such that $\min\left\{d(\sigma^m, \sigma^{m,n_m}), \tilde{d}(\mu^m, \mu^{m,n_m})\right\} < \frac{1}{m}$ (this is possible because G^* is finite). Then σ^{m,n_m} is completely mixed for every $m \in \mathbb{N}$, and $\max\left\{d(\sigma^*, \sigma^{m,n_m}), \tilde{d}(\mu^*, \mu^{m,n_m})\right\} < \max\left\{d(\sigma^*, \sigma^m), \tilde{d}(\mu^*, \mu^m)\right\} + \frac{1}{m}$ (by the triangle inequality), so the fact that $(\sigma^m, \mu^m) \to (\sigma^*, \mu^*)$ implies that $(\sigma^{m,n_m}, \mu^{m,n_m}) \to (\sigma^*, \mu^*)$. Hence, (σ^*, μ^*) is consistent.

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