Chapter 2

Decision Theory

2.1 The basic theory of choice

We consider a set X of alternatives. Alternatives are mutually exclusive in the sense that one cannot choose two distinct alternatives at the same time. We also take the set of feasible alternatives exhaustive so that a player's choices is always well-defined.[1]

We are interested in a player's preferences on X. Such preferences are modeled through a relation $\succeq$ on X, which is simply a subset of $X \times X$. A relation $\succeq$ is said to be *complete* if and only if, given any $x, y \in X$, either $x \succeq y$ or $y \succeq x$. A relation $\succeq$ is said to be *transitive* if and only if, given any $x, y, z \in X$,

$$[x \succeq y \text{ and } y \succeq z] \Rightarrow x \succeq z.$$

A relation is a *preference relation* if and only if it is complete and transitive. Given any preference relation $\succeq$, we can define strict preference $\succ$ by

$$x \succ y \iff [x \succeq y \text{ and } y \not\succeq x],$$

and the indifference $\sim$ by

$$x \sim y \iff [x \succeq y \text{ and } y \succeq x].$$

[1] This is a matter of modeling. For instance, if we have options Coffee and Tea, we define alternatives as $C =$ Coffee but no Tea, $T =$ Tea but no Coffee, $CT =$ Coffee and Tea, and $NT =$ no Coffee and no Tea.

A preference relation can be *represented* by a utility function $u : X \to \mathbb{R}$ in the following sense:

$$x \succeq y \Longleftrightarrow u(x) \geq u(y) \qquad \forall x, y \in X.$$

This statement can be spelled out as follows. First, if $u(x) \geq u(y)$, then the player finds alternative x as good as alternative y. Second, and conversely, if the player finds x at least as good as y, then $u(x)$ must be at least as high as $u(y)$. In other words, the player acts *as if* he is trying to maximize the value of $u(\cdot)$.

The following theorem states further that a relation needs to be a preference relation in order to be represented by a utility function.

Theorem 2.1 *Let X be finite. A relation can be presented by a utility function if and only if it is complete and transitive. Moreover, if $u : X \to \mathbb{R}$ represents $\succeq$, and if $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then $f \circ u$ also represents $\succeq$.*

By the last statement, such utility functions are called ordinal, i.e., only the order information is relevant.

In order to use this ordinal theory of choice, we should know the player's preferences on the alternatives. As we have seen in the previous lecture, in game theory, a player chooses between his strategies, and his preferences on his strategies depend on the strategies played by the other players. Typically, a player does not know which strategies the other players play. Therefore, we need a theory of decision-making under uncertainty.

2.2 Decision-making under uncertainty

Consider a finite set Z of *prizes*, and let P be the set of all probability distributions $p : Z \to [0,1]$ on Z, where $\sum_{z \in Z} p(z) = 1$. We call these probability distributions *lotteries*. A lottery can be depicted by a tree. For example, in Figure 2.1, Lottery 1 depicts a situation in which the player gets \$10 with probability 1/2 (e.g. if a coin toss results in Head) and \$0 with probability 1/2 (e.g. if the coin toss results in Tail).

In the above situation, the probabilities are given, as in a casino, where the probabilities are generated by a machine. In most real-world situations, however, the probabilities are not given to decision makers, who may have an understanding of whether a given event is more likely than another given event. For example, in a game, a player is not

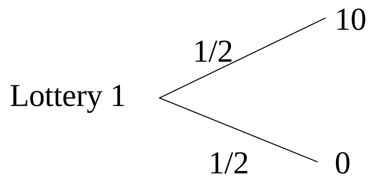


Figure 2.1:

given a probability distribution regarding the other players' strategies. Fortunately, it has been shown by Savage (1954) under certain conditions that a player's beliefs can be represented by a (unique) probability distribution. Using these probabilities, one can represent the decision makers' acts by lotteries.

We would like to have a theory that constructs a player's preferences on the lotteries from his preferences on the prizes. There are many of them. The most well-known—and the most canonical and the most useful—one is the theory of expected utility maximization by Von Neumann and Morgenstern. A preference relation $\succeq$ on P is said to be represented by a von Neumann-Morgenstern utility function $u : Z \to \mathbb{R}$ if and only if

$$p \succeq q \iff U(p) \equiv \sum_{z\in Z} u(z)p(z) \geq \sum_{z\in Z} u(z)q(z) \equiv U(q) \tag{2.1}$$

for each $p, q \in P$. This statement has two crucial parts:

1. $U : P \to \mathbb{R}$ represents $\succeq$ in the ordinal sense. That is, if $U(p) \geq U(q)$, then the player finds lottery p as good as lottery q. And conversely, if the player finds p at least as good as q, then $U(p)$ must be at least as high as $U(q)$.

2. The function U takes a particular form: for each lottery p, $U(p)$ is the expected value of u under p. That is, $U(p) \equiv \sum_{z\in Z} u(z)p(z)$. In other words, the player acts *as if* he wants to maximize *the expected value of* u. For instance, the expected utility of Lottery 1 for the player is $E(u(\text{Lottery 1})) = \frac{1}{2}u(10) + \frac{1}{2}u(0)$.[2]

In the sequel, I will describe the necessary and sufficient conditions for a representation as in (2.1). The first condition states that the relation is indeed a preference relation:

[2] If Z were a continuum, like $\mathbb{R}$, we would compute the expected utility of p by $\int u(z)p(z)dz$.

Figure 2.2: Two lotteries

Axiom 2.1 $\succeq$ *is complete and transitive.*

This is necessary by Theorem 2.1, for U represents $\succeq$ in ordinal sense. The second condition is called *independence* axiom, stating that a player's preference between two lotteries p and q does not change if we toss a coin and give him a fixed lottery r if "tail" comes up.

Axiom 2.2 *For any* $p, q, r \in P$, *and any* $a \in (0, 1]$, $ap + (1 - a)r \succ aq + (1 - a)r \iff p \succ q$.

Let p and q be the lotteries depicted in Figure 2.2. Then, the lotteries $ap+(1-a)r$ and $aq + (1 - a)r$ can be depicted as in Figure 2.3, where we toss a coin between a fixed lottery r and our lotteries p and q. Axiom 2.2 stipulates that the player would not change his mind after the coin toss. Therefore, the independence axiom can be taken as an axiom of "dynamic consistency" in this sense.

The third condition is purely technical, and called *continuity* axiom. It states that there are no "infinitely good" or "infinitely bad" prizes.

Axiom 2.3 *For any* $p, q, r \in P$ *with* $p \succ q$, *there exist* $a, b \in (0, 1)$ *such that* $ap + (1 - a)r \succ q$ *and* $p \succ bq + (1 - b)r$.

Axioms 2.1 and 2.2 imply that, given any $p, q, r \in P$ and any $a \in [0, 1]$,

$$\text{if } p \sim q, \text{ then } ap + (1 - a)\, r \sim aq + (1 - a)r. \tag{2.2}$$

This has two implications:

1. The indifference curves on the lotteries are straight lines.

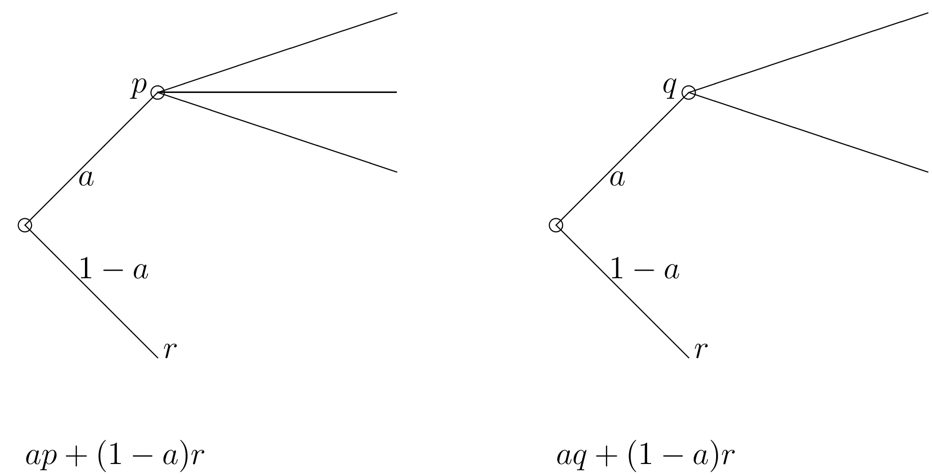


Figure 2.3: Two compound lotteries

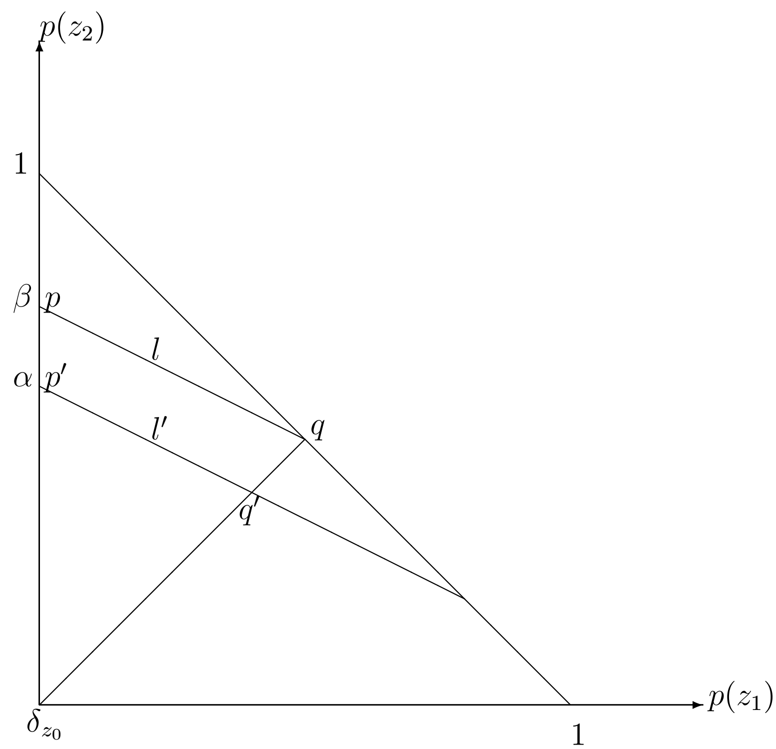


Figure 2.4: Indifference curves on the space of lotteries

2. The indifference curves, which are straight lines, are parallel to each other.

To illustrate these facts, consider three prizes z_0, z_1, and z_2, where $z_2 \succ z_1 \succ z_0$. A lottery p can be depicted on a plane by taking $p(z_1)$ as the first coordinate (on the horizontal axis), and $p(z_2)$ as the second coordinate (on the vertical axis). The remaining probability $p(z_0)$ is $1 - p(z_1) - p(z_2)$. [See Figure 2.4 for the illustration.] Given any two lotteries p and q, the convex combinations $ap + (1-a)\,q$ with $a \in [0,1]$ form the line segment connecting p to q. Now, taking $r = q$, we can deduce from (2.2) that, if $p \sim q$, then $ap + (1-a)\,q \sim aq + (1-a)q = q$ for each $a \in [0,1]$. That is, the line segment connecting p to q is an indifference curve. Moreover, if the lines l and l' are parallel, then $\alpha/\beta = |q'| \,/\, |q|$, where $|q|$ and $|q'|$ are the distances of q and q' to the origin, respectively. Hence, taking $a = \alpha/\beta$, we compute that $p' = ap + (1-a)\,\delta_{z_0}$ and $q' = aq + (1-a)\,\delta_{z_0}$, where δ_{z_0} is the lottery at the origin and gives z_0 with probability 1. Therefore, by (2.2), if l is an indifference curve, l' is also an indifference curve, showing that the indifference curves are parallel.

Line l can be defined by equation $u_1 p(z_1) + u_2 p(z_2) = c$ for some $u_1, u_2, c \in \mathbb{R}$. Since l' is parallel to l, then l' can also be defined by equation $u_1 p(z_1) + u_2 p(z_2) = c'$ for some c'. Since the indifference curves are defined by equality $u_1 p(z_1) + u_2 p(z_2) = c$ for various values of c, the preferences are represented by

$$\begin{aligned} U(p) &= 0 + u_1 p(z_1) + u_2 p(z_2) \\ &\equiv u(z_0)p(z_0) + u(z_1)p(z_1) + u(z_2)p(z_2), \end{aligned}$$

where

$$\begin{aligned} u(z_0) &= 0, \\ u(z_1) &= u_1, \\ u(z_2) &= u_2, \end{aligned}$$

giving the desired representation.

This is true in general, as stated in the next theorem:

Theorem 2.2 *A relation $\succeq$ on P can be represented by a von Neumann-Morgenstern utility function $u : Z \to R$ as in (2.1) if and only if $\succeq$ satisfies Axioms 2.1-2.3. Moreover, u and $\tilde{u}$ represent the same preference relation if and only if $\tilde{u} = au + b$ for some $a > 0$ and $b \in \mathbb{R}$.*

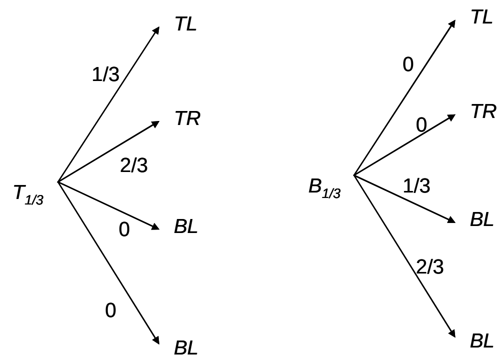
By the last statement in our theorem, this representation is "unique up to affine transformations". That is, a decision maker's preferences do not change when we change his von Neumann-Morgenstern (VNM) utility function by multiplying it with a positive number, or adding a constant to it; but they do change when we transform it through a non-linear transformation. In this sense, this representation is "cardinal". Recall that, in ordinal representation, the preferences wouldn't change even if the transformation were non-linear, so long as it was increasing. For instance, under certainty, $v = \sqrt{u}$ and u would represent the same preference relation, while (when there is uncertainty) the VNM utility function $v = \sqrt{u}$ represents a very different set of preferences on the lotteries than those are represented by u.

2.3 Modeling Strategic Situations

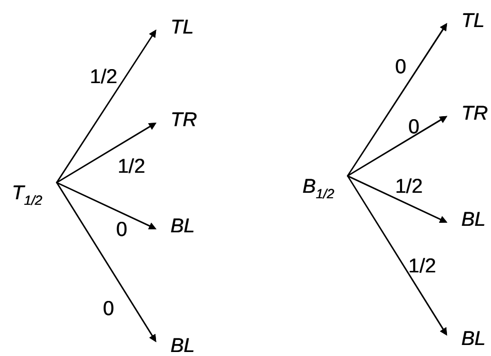
In a game, when a player chooses his strategy, in principle, he does not know what the other players play. That is, he faces uncertainty about the other players' strategies. Hence, in order to define the player's preferences, one needs to define his preference under such uncertainty. In general, this makes modeling a difficult task. Fortunately, using the utility representation above, one can easily describe these preferences in a compact way.

Consider two players Alice and Bob with strategy sets S_A and S_B. If Alice plays s_A and Bob plays s_B, then the outcome is (s_A, s_B). Hence, it suffices to take the set of outcomes $Z = S_A \times S_B = \{(s_A, s_B) | s_A \in S_A, s_B \in S_B\}$ as the set of prizes. Consider Alice. When she chooses her strategy, she has a belief about the strategies of Bob, represented by a probability distribution μ_A on S_B, where $\mu_A(s_B)$ is the probability that Bob plays s_B, for any strategy s_B. Given such a belief, each strategy s_A induces a lottery, which yields the outcome (s_A, s_B) with probability $\mu_A(s_B)$. Therefore, we can consider each of her strategies as a lottery.

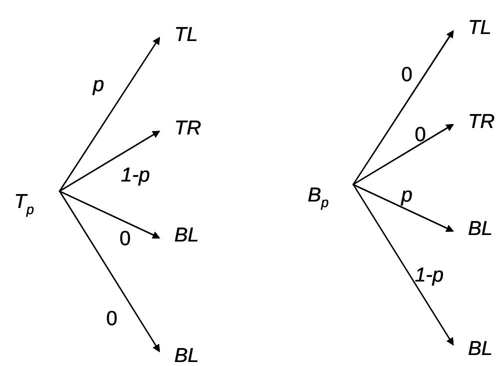
Example 2.1 *Let $S_A = \{T, B\}$ and $S_B = \{L, R\}$. Then, the outcome set is $Z = \{TL, TR, BL, BR\}$. Suppose that Alice assigns probability $\mu_A(L) = 1/3$ to L and $\mu_A(R) = 2/3$ to R. Then, under this belief, her strategies T and B yield the following* *lotteries:*



On the other hand, if she assigns probability $\mu_A(L) = 1/2$ *to* L *and* $\mu_A(R) = 1/2$ *to* R*, then her strategies* T *and* B *yield the following lotteries:*



The objective of a game theoretical analysis is to understand what players believe about the other players' strategies and what they would play. In other words, the players' beliefs, μ_A and μ_B, are determined at the end of the analysis, and we do not know them when we model the situation. Hence, in order to describe a player's preferences, we need to describe his preferences among all the lotteries as above for every possible belief he may hold. In the example above, we need to describe how Alice compares the lotteries



(2.3)

for every $p \in [0,1]$. That is clearly a challenging task.

Fortunately, under Axioms 2.1-2.3, which we will assume throughout the course, we can describe the preferences of Alice by a function

$$u_A : S_A \times S_B \to \mathbb{R}.$$

Similarly, we can describe the preferences of Bob by a function

$$u_B : S_A \times S_B \to \mathbb{R}.$$

In the example above, all we need to do is to find four numbers for each player. The preferences of Alice is described by $u_A(T,L)$, $u_A(T,R)$, $u_A(B,L)$, and $u_A(B,R)$.

Example 2.2 *In the previous example, assume that regarding the lotteries in (2.3), the preference relation of Alice is such that*

$$\begin{aligned} T_p \underset{A}{\succ} B_p &\quad \text{if } p > 1/4 \\ T_p \underset{A}{\sim} B_p &\quad \text{if } p = 1/4 \\ B_p \underset{A}{\succ} T_p &\quad \text{if } p < 1/4, \end{aligned} \tag{2.4}$$

and she is indifferent between the sure outcomes (B,L) and (B,R). Under Axioms 2.1-2.3, we can represent her preferences by

$$\begin{aligned} u_A(T,L) &= 3 \\ u_A(T,R) &= -1 \\ u_A(B,L) &= 0 \\ u_A(B,R) &= 0. \end{aligned}$$

The derivation is as follows. By using the fact that she is indifferent between (B,L) and (B,R), we reckon that $u_A(B,L) = u_A(B,R)$. By the second part of Theorem 2.2, we can set $u_A(B,L) = 0$ (or any other number you like)! Moreover, in (2.3), the lottery T_p yields

$$U_A(T_p) = pu_A(T,L) + (1-p)\,u_A(T,R),$$

and the lottery B_p yields

$$U_A(B_p) = pu_A(B,L) + (1-p)\,u_A(B,R) = 0.$$

Hence, the condition (2.4) can be rewritten as

$$\begin{array}{rcll} pu_A(T,L) + (1-p)\,u_A(T,R) & > & 0 & \text{if } p > 1/4 \\ pu_A(T,L) + (1-p)\,u_A(T,R) & = & 0 & \text{if } p = 1/4 \\ pu_A(T,L) + (1-p)\,u_A(T,R) & < & 0 & \text{if } p < 1/4. \end{array}$$

That is,

$$\frac{1}{4}u_A(T,L) + \frac{3}{4}u_A(T,R) = 0$$

and

$$u_A(T,L) > u_A(T,R).$$

In other words, all we need to do is to find numbers $u_A(T,L) > 0$ *and* $u_A(T,R) < 0$ *with* $u_A(T,L) = -3u_A(T,R)$*, as in our solution. (Why would any such two numbers yield the same preference relation?)*

2.4 Attitudes Towards Risk

Here, we will relate the attitudes of an individual towards risk to the properties of his von-Neumann-Morgenstern utility function. Towards this end, consider the lotteries with monetary prizes and consider a decision maker with utility function $u : \mathbb{R} \to \mathbb{R}$.

A lottery is said to be a *fair* gamble if its expected value is 0. For instance, consider a lottery that gives x with probability p and y with probability $1-p$; denote this lottery by $L(x, y; p)$. Such a lottery is a fair gamble if and only if $px + (1-p)y = 0$.

A decision maker is said to be *risk-neutral* if and only if he is indifferent between accepting and rejecting all fair gambles. Hence, a decision maker with utility function u is risk-neutral if and only if

$$\sum u(x)\,p(x) = u(0) \text{ whenever } \sum xp(x) = 0.$$

This is true if and only if the utility function u is *linear*, i.e., $u(x) = ax + b$ for some real numbers a and b. Therefore, an agent is risk-neutral if and only if he has a linear Von-Neumann-Morgenstern utility function.

A decision maker is *strictly risk-averse* if and only if he rejects *all* fair gambles, except for the gamble that gives 0 with probability 1. That is,

$$\sum u(x)\,p(x) < u(0) = u\left(\sum xp(x)\right).$$

Here, the inequality states that he rejects the lottery p, and the equality is by the fact that the lottery is a fair gamble. As in the case of risk neutrality, it suffices to consider the binary lotteries $L(x, y; p)$, in which case the above inequality reduces to

$$pu(x) + (1 - p)u(y) < u(px + (1 - p)y).$$

This is a familiar inequality from calculus: a function g is said to be *strictly concave* if and only if

$$g(\lambda x + (1 - \lambda)y) > \lambda g(x) + (1 - \lambda)g(y)$$

for all $\lambda \in (0, 1)$. Therefore, strict risk-aversion is equivalent to having a strictly concave utility function. A decision maker is said to be *risk-averse* iff he has a *concave* utility function, i.e., $u(\lambda x + (1 - \lambda)y) \geq \lambda u(x) + (1 - \lambda)u(y)$ for each x, y, and λ. Similarly, a decision maker is said to be (strictly) *risk seeking* iff he has a (strictly) *convex* utility function.

Consider Figure 2.5. A risk averse decision maker's expected utility is $U(gamble) = pu(W_1) + (1 - p)u(W_2)$ if he has a gamble that gives W_1 with probability p and W_2 with probability $1-p$. On the other hand, if he had the expected value $pW_1 + (1 - p)W_2$ for sure, his expected utility would be $u(pW_1 + (1 - p)W_2)$. Hence, the cord AB is the utility difference that this risk-averse agent would lose by taking the gamble instead of its expected value. Likewise, the cord BC is the maximum amount that he is willing to pay in order to avoid taking the gamble instead of its expected value. For example, suppose that W_2 is his wealth level; $W_2 - W_1$ is the value of his house, and p is the probability that the house burns down. In the absence of fire insurance, the expected utility of this individual is EU(gamble), which is lower than the utility of the expected value of the gamble.

2.4.1 Risk sharing

Consider an agent with utility function $u : x \mapsto \sqrt{x}$. He has a (risky) asset that gives \$100 with probability 1/2 and gives \$0 with probability 1/2. The expected utility of the asset for the agent is $EU_0 = \frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{100} = 5$. Consider also another agent who is identical to this one, in the sense that he has the same utility function and an asset that pays \$100 with probability 1/2 and gives \$0 with probability 1/2. Assume throughout that what an asset pays is statistically independent from what the other asset pays. Imagine

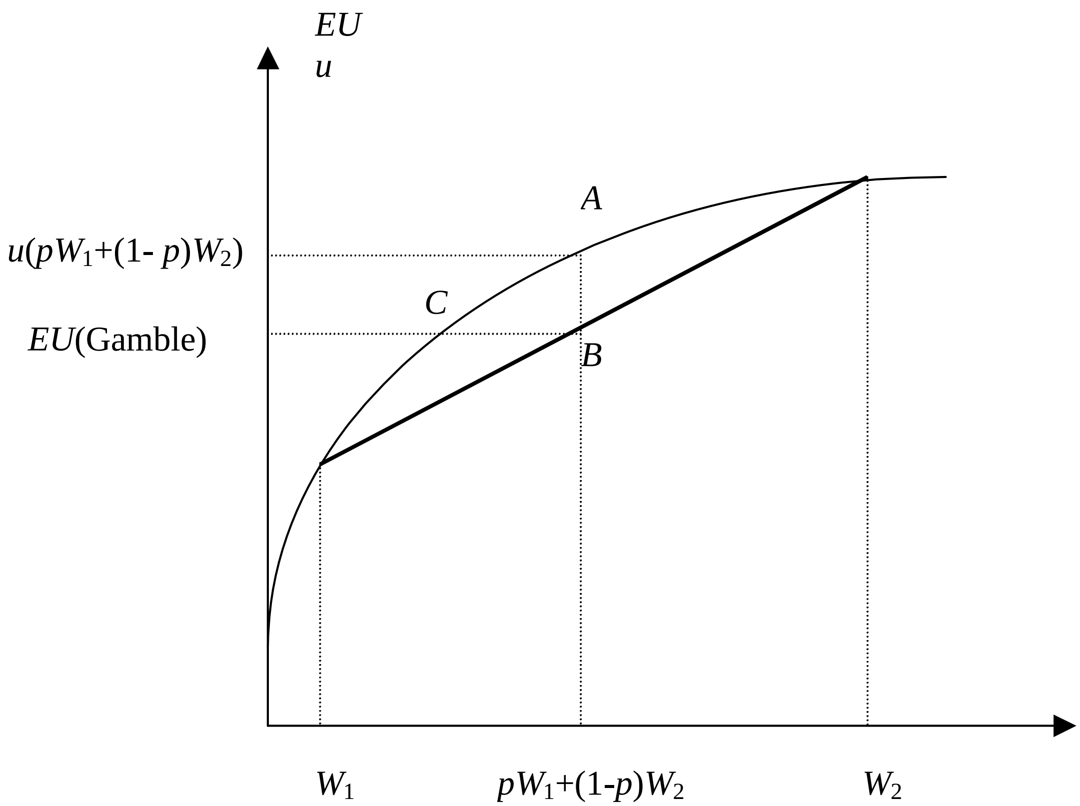


Figure 2.5:

that the two agents form a mutual fund by pooling their assets, each agent owning half of the mutual fund. This mutual fund gives \$200 the probability 1/4 (when both assets yield high dividends), \$100 with probability 1/2 (when only one on the assets gives high dividend), and gives \$0 with probability 1/4 (when both assets yield low dividends). Thus, each agent's share in the mutual fund yields \$100 with probability 1/4, \$50 with probability 1/2, and \$0 with probability 1/4. Therefore, his expected utility from the share in this mutual fund is $EU_S = \frac{1}{4}\sqrt{100} + \frac{1}{2}\sqrt{50} + \frac{1}{4}\sqrt{0} = 6.0355$. This is clearly larger than his expected utility from his own asset which yields only 5. Therefore, the above agents gain from sharing the risk in their assets.

2.4.2 Insurance

Imagine a world where in addition to one of the agents above (with utility function $u : x \mapsto \sqrt{x}$ and a risky asset that gives \$100 with probability 1/2 and gives \$0 with probability 1/2), we have a risk-neutral agent with lots of money. We call this new agent the insurance company. The insurance company can insure the agent's asset, by giving him \$100 if his asset happens to yield \$0. How much premium, P, the agent would be willing to pay to get this insurance? [A premium is an amount that is to be paid to insurance company regardless of the outcome.]

If the risk-averse agent pays premium P and buys the insurance, his wealth will be $\$100 - P$ for sure. If he does not, then his wealth will be \$100 with probability 1/2 and \$0 with probability 1/2. Therefore, he is willing to pay P in order to get the insurance iff

$$u(100 - P) \geq \frac{1}{2}u(0) + \frac{1}{2}u(100)$$

i.e., iff

$$\sqrt{100 - P} \geq \frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{100}.$$

The above inequality is equivalent to

$$P \leq 100 - 25 = 75.$$

That is, he is willing to pay 75 dollars premium for an insurance. On the other hand, if the insurance company sells the insurance for premium P, it gets P for sure and pays \$100 with probability 1/2. Therefore it is willing to take the deal iff

$$P \geq \frac{1}{2}100 = 50.$$

Therefore, both parties would gain, if the insurance company insures the asset for a premium $P \in (50, 75)$, a deal both parties are willing to accept.

Exercise 2.1 *Now consider the case that we have two identical risk-averse agents as above, and the insurance company. Insurance company is to charge the same premium P for each agent, and the risk-averse agents have an option of forming a mutual fund. What is the range of premiums that are acceptable to all parties?*

2.5 Exercises with Solution

1. [Homework 1, 2006] In which of the following pairs of games the players' preferences over lotteries are the same?

(a)

	L	M	R
a	$2,-2$	$1,1$	$-3,7$
b	$1,10$	$0,4$	$0,4$
c	$-2,1$	$1,7$	$-1,-5$

	L	M	R
a	$12,-1$	$5,0$	$-3,2$
b	$5,3$	$3,1$	$3,1$
c	$-1,0$	$5,2$	$1,-2$

(b)

	L	M	R
a	$1,2$	$7,0$	$4,-1$
b	$6,1$	$2,2$	$8,4$
c	$3,-1$	$9,2$	$5,0$

	L	M	R
a	$1,5$	$7,1$	$4,-1$
b	$6,3$	$2,4$	$8,8$
c	$3,-1$	$9,5$	$5,1$

Solution: Recall from Theorem 2.2 that two utility functions represent the same preferences over lotteries if and only if one is an affine transformation of the other. That is, we must have $v_i = \alpha u_i + \beta$ for some α and β where u_i and v_i are the utility functions on the left and right, respectively, for each player i. In Part 1, the preferences of player 1 are different in two games. To see this, note that $u_1(b, M) = 0$ and $v_1(b, M) = 3$. Hence, we must have $\beta = 3$. Moreover, $u_1(c, M) = 1$ and $v_1(c, M) = 5$. Hence, we must have $\alpha = 2$. But then, $\alpha u_1(a, L) + \beta = 7 \neq 12 = v_1(a, L)$, showing that it is impossible to have an affine transformation. Similarly, one can check that the preferences of Player 2 are different in Part 2.

Now, comparisons of payoffs for (c, R) and (b, L) yield that $\alpha = 2$ and $\beta = 1$, but then the payoffs for (b, M) do not match under the resulting transformation.

2. [Homework 1, 2011] Alice and Bob want to meet in one of three places, namely Aquarium (denoted by A), Boston Commons (denoted by B) and a Celtics game (denoted by C). Each of them has strategies A, B, C. If they both play the same strategy, then they meet at the corresponding place, and they end up at different places if their strategies do not match. You are asked to find a pair of utility functions to represent their preferences, assuming that they are expected utility maximizers.

   Alice's preferences: She prefers any meeting to not meeting, and she is indifference towards where they end up if they do not meet. She is indifferent between a situation in which she will meet Bob at A, or B, or C, each with probability $1/3$, and a situation in which she meets Bob at A with probability $1/2$ and does not meet Bob with probability $1/2$. If she believes that Bob goes to Boston Commons with probability p and to the Celtics game with probability $1 - p$, she weakly prefers to go to Boston Commons if and only if $p \geq 1/3$.

   Bob's preferences: If he goes to the Celtics game, he is indifferent where Alice goes. If he goes to Aquarium or Boston commons, then he prefers any meeting to not meeting, and he is indifferent towards where they end up in the case they do not meet. He is indifferent between playing A, B, and C if he believes that Alice may choose any of her strategies with equal probabilities.

   (a) Assuming that they are expected utility maximizers, find a pair of utility functions $u_A : \{A, B, C\}^2 \to \mathbb{R}$ and $u_B : \{A, B, C\}^2 \to \mathbb{R}$ that represent the preferences of Alice and Bob on the lotteries over $\{A, B, C\}^2$.

   Solution: Alice's utility function is determined as follows. Since she is indifferent between any (X, Y) with $X \neq Y$, by Theorem 2.2, one can normalize her payoff for any such strategy profile to $u_A(X, Y) = 0$. Moreover, since she prefers meeting to not meeting, $u_A(X, X) > 0$ for all $X \in \{A, B, C\}$. By Theorem 2.2, one can also set $u_A(C, C) = 1$ by a normalization. The indifference condition in the question can then be written as

$$\frac{1}{3}u_A(A, A) + \frac{1}{3}u_A(B, B) + \frac{1}{3}u_A(C, C) = \frac{1}{2}u_A(A, A).$$

The last preference in the question also leads to

$$\frac{1}{3}u_A(B,B) = \frac{2}{3}u_A(C,C).$$

Form the last equality, $u_A(B,B) = 2$, and from the previous displayed equality, $u_A(A,A) = 6$.

Bob's utility function can be obtained similarly, by setting $u_B(X,Y) = 0$ for any distinct X,Y when $Y \in \{A,B\}$. The first and the last indifference conditions also imply that $u_B(X,C) > 0$, and hence one can set $u_B(X,C) = 1$ for all $X \in \{A,B,C\}$ by the first indifference. The last indifference then implies that

$$\frac{1}{3}u_B(A,A) = \frac{1}{3}u_B(B,B) = u_B(X,C) = 1,$$

yielding $u_B(A,A) = u_B(B,B) = 3$.

(b) Find another representation of the same preferences.

Solution: By Theorem 2.2, we can find another pair of utility functions by doubling all payoffs.

(c) Find a pair of utility functions that yield the same preference as u_A and u_B does among the sure outcomes but do not represent the preferences above.

Solution: Take $u_A(A,A) = 60$ and $u_B(A,A) = u_B(B,B) = 30$ while keeping all other payoffs as before. By Theorem 2.1, the preferences among sure outcomes do not change, but the preferences among some lotteries change by Theorem 2.2.

3. [Homework 1, 2011] In this question you are asked to price a simplified version of mortgage-backed securities. A banker lends money to n homeowners, where each homeowner signs a mortgage contract. According to the mortgage contract, the homeowner is to pay the lender 1 million dollar, but he may go bankrupt with probability p, in which case there will be no payment. There is also an investor who can buy a contract in which case he would receive the payment from the homeowner who has signed the contract. The utility function u of the investor is given by $u(x) = -\exp(-\alpha x)$, where x is the net change in his wealth.

(a) How much is the investor willing to pay for a mortgage contract?

Solution: He pays a price P if and only if $E[u(x-P)] \geq u(0)$, i.e.,

$$-(1-p)\exp(-\alpha(1-P)) - p\exp(-\alpha(0-P)) \geq -1.$$

That is,

$$P \leq P^* \equiv -\frac{1}{\alpha}\ln(p+(1-p)\exp(-\alpha)),$$

where P^* is the maximum willing to pay.

(b) Now suppose that the banker can form n "mortgage-backed securities" by pooling all the mortgage contracts and dividing them equally. A mortgage backed security yields $1/n$ of the total payments by the n homeowners, i.e., if k homeowners go bankrupt, a security pays $(n-k)/n$ millions dollar. Assume that homeowners' bankruptcy are stochastically independent from each other. How much is the investor willing to pay for a mortgage-backed security? Assuming that n is large find an approximate value for the price he is willing to pay. [Hint: for large n, approximately, the average payment is normally distributed with mean $1-p$ (million dollars) and variance $p(1-p)/n$. If X is normally distributed with mean μ and variance σ^2, the expected value of $\exp(-\alpha X)$ is $\exp\left(-\alpha\left(\mu-\frac{1}{2}\alpha\sigma^2\right)\right)$.] How much more can the banker raise by creating mortgage-backed securities? (Use the approximate values for large n.)

Solution: Writing $C_{n,k}$ for the number of k combinations out of n, the probability that there are k bankruptcies is $C_{n,k}p^k(1-p)^{n-k}$. If he pays Q for a mortgage-backed security, his net revenue in the case of k bankruptcies is $1-k/n-Q$. Hence, his expected payoff is

$$-\sum_{k=0}^{n}\exp(-\alpha(1-k/n-Q))\,C_{n,k}p^k(1-p)^{n-k}.$$

He is willing to pay Q if the above amount is at least -1, the payoff from 0. Therefore, he is willing to pay at most

$$Q^* = 1-\frac{1}{\alpha}\ln\left(\sum_{k=0}^{n}\exp(\alpha k/n)\,C_{n,k}p^k(1-p)^{n-k}\right).$$

For large n,

$$Q^* \cong 1-\frac{1}{\alpha}\ln(\exp(\alpha(p+\alpha p(1-p)/(2n)))) = 1-p-\alpha\frac{p(1-p)}{2n}.$$

Note that he is asking a discount of $\alpha p(1-p)/2n$ from the expected payoff against the risk, and behaves approximately risk neutral for large n. The banker gets an extra revenue of $Q^* - P^*$ from creating mortgage-backed securities. (Check that $Q^* - P^* > 0$.)

(c) Answer part (b) by assuming instead that the homeowners' bankruptcy are perfectly correlated: with probability p all homeowners go bankrupt and with probability $1-p$ none of them go bankrupt. Briefly compare your answers for parts (b) and (c).

Solution: With perfect correlation, a mortgage-backed security is equivalent to one contract, and hence he is willing to pay at most P^*. In general, when there is a positive correlation between the bankruptcies of different homeowners (e.g. due to macroeconomic conditions), the value of mortgage backed securities will be less than what it would have been under independence. Therefore, mortgage back securities that are priced under the erroneous assumption of independence would be over-priced.

2.6 Exercises

1. [Homework 1, 2000] Consider a decision maker with Von Neumann and Morgenstren utility function u with $u(x) = (x-1)^2$. Check whether the following VNM utility functions can represent this decision maker's preferences. (Provide the details.)

   (a) $u^* : x \mapsto x - 1$;

   (b) $u^{**} : x \mapsto (x-1)^4$;

   (c) $\hat{u} : x \mapsto -(x-1)^2$;

   (d) $\tilde{u} : x \mapsto 2(x-1)^2 - 1$.

2. [Homework 1, 2004] Which of the following pairs of games are strategically equivalent, i.e., can be taken as two different representations of the same decision problem?

(a)

	L	R
T	2,2	4,0
B	3,3	1,0

	L	R
T	-6,4	0,0
B	-3,6	-9,0

(b)

	L	R
T	2,2	4,0
B	3,3	1,0

	L	R
T	4,4	16,0
B	9,9	1,0

(c)

	L	R
T	2,2	4,0
B	3,3	1,0

	L	R
T	4,2	2,0
B	3,3	1,0

3. [Homework 1, 2001] We have two dates: 0 and 1. We have a security that pays a single dividend, at date 1. The dividend may be either \$100, or \$50, or \$0, each with probability 1/3. Finally, we have a risk-neutral agent with a lot of money. (The agent will learn the amount of the dividend at the beginning of date 1.)

   (a) An agent is asked to decide whether to buy the security or not at date 0. If he decides to buy, he needs to pay for the security only at date 1 (not immediately at date 0). What is the highest price π_S at which the risk-neutral agent is willing to buy this security?

   (b) Now consider an "option" that gives the holder the right (but not obligation) to buy this security at a strike price K at date 1 — after the agent learns the amount of the dividend. If the agent buys this option, what would be the agent's utility as a function of the amount of the dividend?

   (c) An agent is asked to decide whether to buy this option or not at date 0. If he decides to buy, he needs to pay for the option only at date 1 (not immediately at date 0). What is the highest price π_O at which the risk-neutral agent is willing to buy this option?

4. [Homework 1, 2001] Take $X = \mathbb{R}$, the set of real numbers, as the set of alternatives. Define a relation $\succeq$ on X by

$$x \succeq y \iff x \geq y - 1/2 \qquad \text{for all } x, y \in X.$$

(a) Is $\succeq$ a preferece relation? (Provide a proof.)

(b) Define the relations $\succ$ and $\sim$ by

$$x \succ y \iff [x \succeq y \text{ and } y \not\succeq x]$$

and

$$x \sim y \iff [x \succeq y \text{ and } y \succeq x],$$

respectively. Is $\succ$ transitive? Is $\sim$ transitive? Prove your claims.

(c) Would $\succeq$ be a preference relation if we had $X = \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of all natural numbers?