Econ 14.04 Fall 2006
Solutions to Assignment 2: Indirect Utility, Expenditure Functions, and Duality

1. Setting up the KT we have:

$$f(x,\alpha,\beta)+\lambda g(x,\alpha,\beta)$$

Taking the FOC for each x_i we have:

$$\frac{\partial f}{\partial x_i}+\lambda\frac{\partial g}{\partial x_i}\equiv 0$$

Notice that these are identities - ie they always hold. Suppose that we are at the optimal $x^*(a,\beta)$. If we take the derivative of M with respect to α we get:

$$\frac{\partial M}{\partial \alpha}=\frac{\partial f}{\partial \alpha}+\lambda\frac{\partial g}{\partial \alpha}+\sum_{i=1}^{n}(\frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial \alpha}+\lambda\frac{\partial g}{\partial x_i}\frac{\partial x_i}{\partial \alpha})$$

Using the FOC above and noting that we can factor $\frac{\partial x_i}{\partial \alpha}$ out of each summation term, all the last terms cancel out and we are left with:

$$\frac{\partial M}{\partial \alpha}=\frac{\partial f}{\partial \alpha}+\lambda\frac{\partial g}{\partial \alpha}$$

The same thing holds for β.

2. For each of the following, derive $\mathbf{x}(\mathbf{p},m), \mathbf{e}(\mathbf{p},u), v(\mathbf{p},m), \mathbf{h}(\mathbf{p},u)$ using the standard budget constraint $p_1x_1+p_2x_2=m$:

   (a) The utility function here is strictly concave. Since the budget constraint is linear, we will always end up at a corner. Thus:

$$\begin{aligned}\mathbf{x}_1(\mathbf{p},m) &= \begin{matrix}\frac{m}{p_1} & p_1\le p_2\\ 0 & otherwise\end{matrix}\\ \mathbf{x}_2(\mathbf{p},m) &= \begin{matrix}\frac{m}{p_1} & p_1> p_2\\ 0 & otherwise\end{matrix}\\ v(\mathbf{p},m) &= \frac{m}{\min(p_1,p_2)}\end{aligned}$$

   $v(\mathbf{p},e(p,u))=u\rightarrow\frac{e(p,u)}{\min(p_1,p_2)}=u$ so:

$$\begin{aligned}e(\mathbf{p},m) &= u\min(p_1,p_2)\\ h_1(p,u) &= \begin{matrix}u & p_1\le p_2\\ 0 & otherwise\end{matrix}\\ h_2(p,u) &= \begin{matrix}u & p_1> p_2\\ 0 & otherwise\end{matrix}\end{aligned}$$

   (b) $u(x_1,x_2)=\min(x_1,x_2)$

(c) First argue that the only point that will ever be chosen has $x_1 = x_2$:

1. Suppose that $x_1 > x_2$, $\exists\varepsilon$ such that $u(x_1 - \varepsilon, x_2 + \frac{p_2}{p_1}\varepsilon) > u(x_1, x_2)$. but $p_1(x_1 - \varepsilon) + p_2(x_2 + \varepsilon) = m$ and is affordable. Thus the agent is not profit maximizing.
2. Now substitute in for x_2 and solve the simplified problem:

$$\begin{aligned} &\max x_1 \\ ST \ &: \ (p_1 + p_2)x_1 = m \end{aligned}$$

3. Solving yields:

$$\begin{aligned} x_1(\mathbf{p},m) &= x_2(\mathbf{p},m) = \frac{m}{p_1 + p_2} \\ v(\mathbf{p},m) &= \frac{m}{p_1 + p_2} \\ e(\mathbf{p},u) &= u(p_1 + p_2) \\ h_1(\mathbf{p},u) &= h_1(\mathbf{p},u) = u \end{aligned}$$

(d) This problem is done in recitation notes one with minor alterations:

$$\begin{aligned} \mathbf{x}_1(\mathbf{p},m) &= \begin{matrix} \frac{m}{p_1} & \frac{p_1}{2} \leq p_2 \\ 0 & otherwise \end{matrix} \\ \mathbf{x}_2(\mathbf{p},m) &= \begin{matrix} \frac{m}{p_1} & \frac{p_1}{2} > p_2 \\ 0 & otherwise \end{matrix} \\ v(\mathbf{p},m) &= \frac{m}{\min(\frac{p_1}{2}, p_2)} \end{aligned}$$

$$\begin{aligned} e(\mathbf{p},m) &= u \min(\frac{p_1}{2}, p_2) \\ h_1(p,u) &= \begin{matrix} \frac{u}{2} & \frac{p_1}{2} \leq p_2 \\ 0 & otherwise \end{matrix} \\ h_2(p,u) &= \begin{matrix} u & \frac{p_1}{2} > p_2 \\ 0 & otherwise \end{matrix} \end{aligned}$$

Note that this solution is identical to the max(x_1, x_2) solution with the slight difference that when the $\frac{p_1}{2} = p_2$ any combination of inputs yield the same solution.

(e) Taking the FOC:

$$\begin{aligned} (1) &: \frac{1}{2}x_1^{-\frac{1}{2}}x_2^{\frac{1}{3}} - \lambda p_1 = 0 \\ (2) &: \frac{1}{3}x_1^{\frac{1}{2}}x_2^{-\frac{2}{3}} - \lambda p_2 = 0 \\ (3) &: p_1x_1 + p_2x_2 = 0 \end{aligned}$$

Dividing (1) by (2):

$$\frac{3}{2}\frac{x_2}{x_1} = \frac{p_1}{p_2} \rightarrow x_2 = \frac{p_1}{p_2}\frac{2}{3}x_1$$

Substitution into (3) yields:

$$p_1x_1 + p_1\frac{2}{3}x_1 = m$$

Thus:

$$\begin{aligned}
\mathbf{x}_1(\mathbf{p},m) &= \frac{3}{5}\frac{m}{p_1} \\
\mathbf{x}_2(\mathbf{p},m) &= \frac{2}{5}\frac{m}{p_2} \\
v(\mathbf{p},m) &= \left(\frac{3}{5}\frac{m}{p_1}\right)^{1/2}\left(\frac{2}{5}\frac{m}{p_2}\right)^{1/3} = \left(\frac{m}{5}\right)^{\frac{5}{6}}\left(\frac{3}{p_1}\right)^{1/2}\left(\frac{2}{p_2}\right)^{1/3}
\end{aligned}$$

Inverting to get the expenditure function we have:

$$u = \left(\frac{e(p,u)}{5}\right)^{\frac{5}{6}}\left(\frac{3}{p_1}\right)^{1/2}\left(\frac{2}{p_1}\right)^{1/3} \rightarrow e(p,u) = 5u^{\frac{6}{5}}\left(\frac{p_1}{3}\right)^{\frac{3}{5}}\left(\frac{p_2}{2}\right)^{\frac{2}{5}}$$

Taking the derivatives wrt p_1 and p_2 yields the hicksian demands:

$$\begin{aligned}
h_1(p,u) &= 3u^{\frac{6}{5}}\left(\frac{p_1}{3}\right)^{-\frac{2}{5}}\left(\frac{p_2}{2}\right)^{\frac{2}{5}} \\
h_1(p,u) &= 2u^{\frac{6}{5}}\left(\frac{p_1}{3}\right)^{\frac{3}{5}}\left(\frac{p_2}{2}\right)^{-\frac{3}{5}}
\end{aligned}$$

(f) The point of this exercise is to note that we get an identical outcome to problem e. The solution concept is the same.

3. Assuming free disposal we have:

$$u(l,t,g) = \min(l^2, g^2 + t^2)$$

As we saw in problem 2, a min function requires that the two sides be equal and a linear function requires us to use the imput that is cheapest. Thus when $p_g > p_t$, we have $l^2 = t^2, g = 0$. We thus spend $\frac{1}{2}$ our budget on lime and tonic yielding:

$$\begin{aligned}
l(p,m) &= \left(\frac{m}{(p_l + \min(p_g,p_t))}\right)^{\frac{1}{2}} \\
g_1(\mathbf{p},m) &= \begin{matrix} \left(\frac{m}{(p_l+\min(p_g,p_t))}\right)^{\frac{1}{2}} & p_g \leq p_t \\ 0 & otherwise \end{matrix} \\
t_2(\mathbf{p},m) &= \begin{matrix} \left(\frac{m}{(p_l+\min(p_g,p_t))}\right)^{\frac{1}{2}} & p_g > p_t \\ 0 & otherwise \end{matrix} \\
v(\mathbf{p},m) &= \left(\frac{m}{(p_l + \min(p_g,p_t))}\right)^{\frac{1}{2}}
\end{aligned}$$

4. Problem 1:

   (a) Utility functions are ordinal - this allows us to take monotonic transformations without changing the underlying demand functions.

   (b) Transforming the data we have $\widetilde{U}(x_1, x_2) = [\ln(U)]^3 = x_1 + \ln(x_2)$
   The $\text{MRS}_{12} = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = x_2$. Since for $x_1 = 0$, this is a non infinite amount, we may have the case that $x_1 = 0$.

   The $\text{MRS}_{21} = \frac{\frac{\partial U}{\partial x_2}}{\frac{\partial U}{\partial x_1}} = \frac{1}{x_2}$. Since for $x_2 = 0$,$\text{MRS}_{21} = \infty$, we will never use zero of input 2.

   (c) $\text{Max}_{x_1,x_2}\ \widetilde{U}(x_1, x_2)$ st $p_1x_1 + p_2x_2 = M$, $x_1 \geq 0$

$$L : x_1 + \ln(x_2) + \lambda(M - p_1x_1 - p_2x_2) + \mu x_1$$

   FOC:

$$\begin{aligned} 1 + \mu &= \lambda p_1 \\ \frac{1}{x_2} &= \lambda p_2 \\ p_1x_1 + p_2x_2 &= M \\ x_1 &\geq 0, \mu \geq, x_1\mu = 0 \end{aligned}$$

   Eliminating λ we have:

$$x_2(1 + \mu) = \frac{p_1}{p_2}$$

   so when $x_1 > 0$, $x_2 = \frac{p_1}{p_2}, x_1 = \frac{M}{p_1} - 1$. This will only occur if $M > p_1$
   when $\mu > 0$ ($x_1 = 0$), $x_2 = \frac{M}{p_2}$ by the budget constraint.

   We thus have:

$$x_1(p_1, p_2, m) = \begin{matrix} 0 & m < p_1 \\ \frac{m}{p_1} - 1 & otherwise \end{matrix}$$

$$x_2(p_1, p_2, m) = \begin{matrix} \frac{m}{p_2} & m < p_1 \\ \frac{p_1}{p_2} & otherwise \end{matrix}$$

$$V(p_1, p_2, m) = \begin{matrix} e^{[\ln(\frac{m}{p_2})]^{\frac{1}{3}}} & m < p_1 \\ e^{[\frac{m}{p_1} - 1 + \ln(\frac{p_1}{p_2})]^{\frac{1}{3}}} & otherwise \end{matrix}$$

   (d) The expenditure function $v(p, e(p, u)) = u$. Thus, after some rearranging we have:

$$e(p, u) = \begin{matrix} [\ln(u)]^3 p_2 & m < p_1 \\ \left\{[\ln(u)]^3 + 1 - \ln(\frac{p_1}{p_2})\right\} p_1 & otherwise \end{matrix}$$

5. Consider the indirect utility function given by:

$$v(p_1, p_2, m) = \frac{m}{p_1 + p_2}$$

(a) $x_1(p,m) = -\frac{\frac{\partial v}{\partial p_i}}{\frac{\partial v}{\partial m}} = -\frac{-\frac{m}{(p_1+p_2)^2}}{\frac{1}{(p_1+p_2)}} = \frac{m}{p_1+p_2}$. Thus:

$$x_1(p,m) = x_2(p,m) = \frac{m}{p_1+p_2}$$

(b) $v(p_1,p_2,m) = \frac{m}{p_1+p_2} \rightarrow u = v(p_1,p_2,e(p,u)) = \frac{e(p,u)}{p_1+p_2}$. Thus:

$$e(p,u) = u(p_1+p_2)$$

(c) To find a representation of the utility function we solve:

$$\min_{p_1,p_2} \frac{m}{p_1+p_2}$$
$$ST \; : \; x_1p_1 + x_2p_2 = m$$

The FOC are:

$$(1) \; : \; -\frac{m}{(p_1+p_2)^2} + \lambda x_1 = 0$$
$$(2) \; : \; -\frac{m}{(p_1+p_2)^2} + \lambda x_2 = 0$$
$$(3) \; : \; x_1p_1 + x_2p_2 = m$$

From (1) and (2), $x_1 = x_2$. Thus one utility function that can satisfy this is u($x_1, x_2) = \min(x_1, x_2)$

6. *Consider the utility function:

$$u(x_1,x_2) = \min(2x_1 + x_2, x_1 + 2x_2)$$

(a) The indifference curve will be the NE boundary of the two lines.

(b) The slope of a budget line is $-\frac{p_1}{p_2}$. If the budget line is steeper than 2, $x_1 = 0$. Thus $x_1 = 0$ if $\frac{p_1}{p_2} > 2$,

(c) This is identical to b: if $\frac{p_1}{p_2} < \frac{1}{2}$, $x_2 = 0$

(d) If the optimum is unique and on the interior, it must be that $x_1+2x_2 = 2x_1+x_2 \rightarrow x_1 = x_2 \rightarrow \frac{x_1}{x_2} = 1$.

(a) Suppose you have no data:
   1. uncomparable
   2. Bundle 1 $\succsim$ Bundle 2

(b) Suppose that you observe that when $p_1 = 1, p_2 = 1, m = 10$ the consumer chooses $x_1 = 2, x_2 = 8$
   1. Bundle 1 $\precsim$ Bundle 2
   2. Bundle 1 $\precsim$ Bundle 2 (note that this one isn't strict)

(c) Suppose that we have two observations. When $p_1 = 1, p_2 = 1, m = 10$ the consumer chooses $x_1 = 2, x_2 = 8$. When $p_1 = 1, p_2 = 3, m = 15$ the consumer chooses $x_1 = 15, x_2 = 0$
   1. Bundle 1 $\succsim$ Bundle 2
   2. uncomparable