

# Lecture 24

## Parameterizing Eddies

In Gent and McWilliams, 1990, a parameterization of sub-grid scale eddies (geostrophic) eddies was proposed that built on the idea sub-grid scale processes can transport properties as well as diffuse and that baroclinic instability removed large scale potential energy.

### 11.1 Along isopycnal fluxes

Ignoring the zonal direction for convenience, the Reynolds averaged buoyancy equation integrated by a coarse resolution model takes the form

$$\partial_t \bar{b} + \partial_y \bar{v} \bar{b} + \partial_z \bar{w} \bar{b} = -\partial_y \overline{v'b'} - \partial_z \overline{w'b'}$$

where the fluxes  $\overline{v'b'}$  and  $\overline{w'b'}$  are the Reynolds averaged fluxes that represent the interaction of the sub-grid scale processes with the large scale (resolved) model.

If we only consider eddy-fluxes to be oriented along isopycnals then

$$\begin{pmatrix} \overline{v'b'} \\ \overline{w'b'} \end{pmatrix} \cdot \begin{pmatrix} \partial_y \bar{b} \\ \partial_z \bar{b} \end{pmatrix} = \overline{v'b'} \partial_y \bar{b} + \overline{w'b'} \partial_z \bar{b} = 0$$

which means we can eliminate one of the components of the eddy flux in favor of the other:

$$\overline{w'b'} = -\frac{\partial_y \bar{b}}{\partial_z \bar{b}} \overline{v'b'} = s_\rho \overline{v'b'}$$

where  $s_\rho = -\partial_y \bar{b} / \partial_z \bar{b}$  is the isopycnal slope.

The divergence of eddy fluxes can then be written

$$\begin{aligned}
\partial_y \overline{v'b'} + \partial_z \overline{w'b'} &= \partial_y \overline{v'b'} + \partial_z (s_\rho \overline{v'b'}) \\
&= \partial_y \left( \frac{\overline{v'b'}}{\partial_z \bar{b}} \partial_z \bar{b} \right) - \partial_z \left( \frac{\overline{v'b'}}{\partial_z \bar{b}} \partial_y \bar{b} \right) \\
&= \frac{\overline{v'b'}}{\partial_z \bar{b}} \partial_{yz} \bar{b} + \partial_y \left( \frac{\overline{v'b'}}{\partial_z \bar{b}} \right) \partial_z \bar{b} - \partial_z \left( \frac{\overline{v'b'}}{\partial_z \bar{b}} \right) \partial_y \bar{b} - \frac{\overline{v'b'}}{\partial_z \bar{b}} \partial_{yz} \bar{b} \\
&= \partial_y \left( \frac{\overline{v'b'}}{\partial_z \bar{b}} \right) \partial_z \bar{b} - \partial_z \left( \frac{\overline{v'b'}}{\partial_z \bar{b}} \right) \partial_y \bar{b} \\
&= \partial_y \psi^* \partial_z \bar{b} - \partial_z \psi^* \partial_y \bar{b} \\
&= v^* \partial_z \bar{b} + w^* \partial_y \bar{b}
\end{aligned}$$

where  $v^*$  and  $w^*$  are components of a “bolus” flow corresponding to an eddy-induced stream function,  $\psi^* = \overline{v'b'}/\partial_z \bar{b}$ .

## 11.2 Gent-McWilliams

GM90 proposed a closure that assumes that the eddies remove potential energy from the mean flow. The potential energy equation corresponding to the large scale flow is (approximately)

$$\partial_t(z\bar{b}) + \partial_y(v\bar{b}z) + \partial_z(w\bar{b}z) - w\bar{b} = -\partial_y(\overline{v'b'z}) - \partial_z(\overline{w'b'z}) + \overline{w'b'}$$

so to ensure that the parameterized eddies reduce the mean potential energy (in the volume integral as well as locally) we require

$$\overline{w'b'} = \frac{\partial_y \bar{b}}{\partial_z \bar{b}} \overline{v'b'} \leq 0$$

Assuming that the flow is stably stratified ( $\partial_z \bar{b} > 0$ ) then the simplest closure that satisfies the sign constraint is

$$\overline{v'b'} = -\kappa \partial_y \bar{b} \quad \text{so that} \quad \overline{w'b'} = -\frac{\kappa \partial_y \bar{b} \partial_y \bar{b}}{\partial_z \bar{b}}$$

and the bolus transport function is proportional to the isopycnal slope (or baroclinicity):

$$\psi^* = -\kappa \frac{\partial_y \bar{b}}{\partial_z \bar{b}} = \kappa s_\rho$$

### 11.2.1 Other tracers

So far we have worked with the dynamical scalar buoyancy,  $b$ . The Reynolds average equation for an arbitrary tracer,  $\tau$ , is

$$\partial_t \bar{\tau} + \partial_y \bar{v} \bar{\tau} + \partial_z \bar{w} \bar{\tau} = -\partial_y \overline{v' \tau'} - \partial_z \overline{w' \tau'}$$

If we make the same assumption that eddy fluxes are oriented along isopycnals we can try to follow the above procedure of expressing the divergence of eddy tracer fluxes as a transport:

$$\begin{aligned} \partial_y \overline{v' \tau'} + \partial_z \overline{w' \tau'} &= \partial_y \overline{v' \tau'} + \partial_z (s_\rho \overline{v' \tau'}) \\ &= \partial_y \left( \frac{\overline{v' \tau'}}{\partial_z \bar{b}} \right) \partial_z \bar{b} - \partial_z \left( \frac{\overline{v' \tau'}}{\partial_z \bar{b}} \right) \partial_y \bar{b} \end{aligned}$$

which is not as easy to interpret.

However, since we are considering the adiabatic limit, with fluxes oriented only along the isopycnals, and since the buoyancy equation can be written in a Lagrangian form

$$\partial_t \bar{b} + (\bar{v} + v^*) \partial_y \bar{b} + (\bar{w} + w^*) \partial_z \bar{b} = 0$$

where the transporting flow is the residual circulation, then particles of fluid conserve their buoyancy following the modified motion,  $\vec{v} + \vec{v}^*$ . Each particle can be labelled by a tracer value which, in the absence of sources and mixing, means we can immediately write

$$\partial_t \bar{\tau} + (\bar{v} + v^*) \partial_y \bar{\tau} + (\bar{w} + w^*) \partial_z \bar{\tau} = 0$$

and see that in the adiabatic limit tracers must be transported by the same residual flow that transports buoyancy.

## 11.3 Along isopycnal mixing (Redi)

GM90 also suppose that SGS eddies, as well as transport, tend to mix properties along isopycnals. Redi, 19??, had already given the method for rotating the diffusive flux so that it is oriented along isopycnal. The rotated diffusive flux is written

$$F_R = -\kappa_R \underline{\underline{\mathbf{K}}}_R \nabla \bar{b} = -\kappa_R \frac{1}{|\nabla \bar{b}|^2} \begin{pmatrix} \partial_z b^2 & -\partial_y \bar{b} \partial_z \bar{b} \\ -\partial_y \bar{b} \partial_z \bar{b} & \partial_y b^2 \end{pmatrix} \nabla \bar{b}$$

where  $\underline{\underline{\mathbf{K}}}_R$  is a rotation tensor.

Note that  $\underline{\underline{\mathbf{K}}}_R \nabla \bar{b} = 0$  so the buoyancy equation is (obviously) unaffected by along isopycnal mixing.

For stratified conditions in the ocean,  $|\partial_y \bar{b}| \ll |\partial_z \bar{b}|$  so that the isopycnal slopes are small  $|s_\rho| \ll 1$ . The Redi tensor then can be approximated as

$$\underline{\underline{\mathbf{K}}}_R \approx \frac{1}{|\partial_z \bar{b}|^2} \begin{pmatrix} 1 & -\partial_y \bar{b} \partial_z \bar{b} \\ -\partial_y \bar{b} \partial_z \bar{b} & \partial_y \bar{b}^2 \end{pmatrix} = \begin{pmatrix} 1 & s_\rho \\ s_\rho & s_\rho^2 \end{pmatrix}$$

This is known as the small slope approximation to the Redi mixing tensor and has been widely used in ocean models for convenience.

## 11.4 Skew flux

Griffies, 199?, pointed out that an advective flux can be written as a skew diffusive flux as follows:

$$\begin{aligned} \partial_y(v^* \bar{\tau}) + \partial_z(w^* \bar{\tau}) &= \partial_y((- \partial_z \psi^*) \bar{\tau}) + \partial_z((\partial_y \psi^*) \bar{\tau}) \\ &= (\partial_z \psi^*) \partial_y \bar{\tau} - (\partial_y \psi^*) \partial_z \bar{\tau} \\ &= \partial_z(\psi^* \partial_y \bar{\tau}) - \partial_y(\psi^* \partial_z \bar{\tau}) \\ &= \nabla \cdot \left[ \begin{pmatrix} 0 & -\psi^* \\ \psi^* & 0 \end{pmatrix} \begin{pmatrix} \partial_y \bar{\tau} \\ \partial_z \bar{\tau} \end{pmatrix} \right] \end{aligned}$$

Using the GM90 closure ( $\psi^* = \kappa s_\rho$ ) then the GM90 parameterized eddy flux is

$$F_{GM} = \kappa \begin{pmatrix} 0 & -s_\rho \\ s_\rho & 0 \end{pmatrix} \begin{pmatrix} \partial_y \bar{\tau} \\ \partial_z \bar{\tau} \end{pmatrix} = \kappa \underline{\underline{\mathbf{K}}}_{GM} \nabla \bar{\tau}$$

If we use the GM90 scheme in conjunction with small slope approximation to Redi mixing we get the net parameterized flux

$$F_R + F_{GM} = \begin{pmatrix} \kappa_R & (\kappa_R - \kappa_{GM}) s_\rho \\ (\kappa_R + \kappa_{GM}) s_\rho & \kappa_R s_\rho^2 \end{pmatrix} \nabla \bar{\tau}$$

Griffies noted that using the small slope approximation that there is a tendency for natural cancellation between some components of the flux; there is no strong argument for whether  $\kappa_R = \kappa_{GM}$  or not but in ocean models we typically assume the cancellation is perfect.

## 11.5 Transformed Eulerian Mean (TEM)

The TEM approach is to realize that parameterizing the eddy induced velocity is not any easier than parameterizing any other component of momentum: nonlocal effects due to pressure gradients make local parameterizations for momentum very suspicious. Thus the TEM strategy is to make the residual velocity a prognostic variable and write the equation for  $u^*$ . The momentum equation for  $u^*$  is not closed of course (otherwise we would all be happy relaxing at the beach), and it involves eddy fluxes. The advantage is that in the limit of small Ro, the only eddy term appearing in the equation for  $u^*$  is the along-isopycnal flux of Ertel PV. Since we believe that along-isopycnal fluxes are directed down the mean isopycnal gradients (see Redi), we are in business. The QG zonal case is illuminating,

$$\partial_t u^* - f v_a = \overline{v'q'} = -kq_y = -k(-u_y + f/N^2 b_z)_y$$

so that we see that by using a Redi closure for the isopycnal (= horizontal in QG) PV flux we have closed the momentum equation. The closure is GM-like if  $k=\text{const}$ ,  $f=\text{const}$ ,  $N^2$  does not depend on  $z$ .