1 Introduction

We describe here the quasi-geostrophic ocean model that was developed by John Marshall, George Nurser and Roger Brugge - see Marshall, J., A.J.G. Nurser and R. Brugge: (1988) On the time-averaged flow of quasi-geostrophic wind-driven gyres, J. Geophys. Res. (Oceans), 93, 15427-15436. It integrates a prognostic (potential vorticity) equation for each layer in a closed basin with (linearized) bottom topography and can be driven by both wind and buoyancy forces. The model can be integrated in complex geometries; basin and channel versions are available. Both wind and buoyancy driven flows can be studied, but within the confines of quasi-geostrophic dynamics.

2 The model

The multi-layer quasi-geostrophic (Q-G) basin model integrates the potential vorticity equation

$$\overline{\frac{\partial q_n}{\partial t}} + J(\psi_n, q_n) = G_n \tag{1}$$

in each layer n, where the Jacobian $J(\alpha, \beta)$ is given by

$$J(\alpha, \beta) = \overline{\frac{\partial \alpha}{\partial x}}\,\overline{\frac{\partial \beta}{\partial y}} - \overline{\frac{\partial \alpha}{\partial y}}\,\overline{\frac{\partial \beta}{\partial x}}. \tag{2}$$

It functions as follows:

(I) Given the Q-G potential vorticity q and streamfunction ψ in each layer and known forcing functions G_n, the tendency field $\partial q/\partial t$ is calculated and used to update the q field.

(II) This revised q field is inverted with appropriate boundary conditions to give the new streamfunction field

(III) These new q and ψ fields are then used to calculate the tendency in (I) and so on.

The equations are finite-differenced on a regular grid using standard second order finite differences and are stepped forward in time using a leap-frog scheme.

Boundary conditions upon the baroclinic modes required for the elliptic problem are deduced from the constraint that the total quantity of fluid in each layer is known from the continuity equation. A high order $\nabla^6\psi$ (enstrophy-destroying) friction is employed demanding two extra boundary conditions. We have chosen to apply the boundary conditions $\nabla^2\psi = \nabla^4\psi = 0$. Other boundary conditions (for example no-slip) can easily be introduced.

Before going on to describe the numerical implementation we discuss the dynamical set-up, the potential vorticity equation and its non-dimensionalisation.

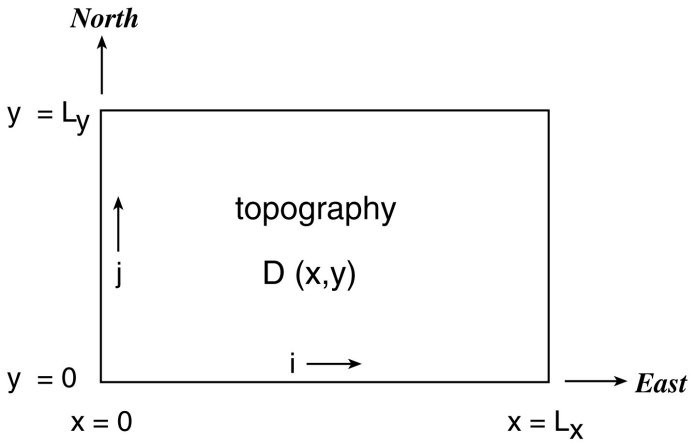

Figure 2.1: The ocean basin and its topography. (See text for meaning of symbols.)

2.1 The physics of the model

The equations are solved in an $L_x \times L_y$ rectangular basin (typically $L_x, L_y = 3000$ km) with bottom topography $D(x, y)$, see Fig. 2.1. The ocean is taken to have total depth H (typically 5000 m) with N layers of thicknesses $H_1, \ldots, H_n, \ldots, H_N$, (see Fig. 2.2) and densities $\rho_1, \ldots, \rho_n, \ldots, \rho_N$. We denote the density jumps $\rho_{n+1,n} = \rho_{n+1} - \rho_n$ between the layers.

The $\eta_{n+1,n}$, the displacements from equilibrium of the interfaces between the $(n+1)$th and nth layer, and also the bottom topography $D(x, y)$ are required [for consistency with the quasi-geostrophic scaling] to be small relative to the layer thicknesses. Note that the double suffix e.g. $\eta_{n+1,n}$ denotes a quantity evaluated at the interfaces between layers $(n+1)$ and n — the 'thermodynamic levels' in the equivalent level formulation.

The Q-G potential vorticity equation may be derived from:

1) the vorticity equation for each layer

$$\frac{D}{Dt}(\nabla^2\psi + \beta y) = f_0 \frac{\partial w}{\partial z} \tag{3}$$

which in the nth layer takes the finite difference form

$$\frac{D}{Dt}(\nabla^2\psi_n + \beta y) = \frac{f_0}{H_n}(w_{n,n-1} - w_{n+1,n}) \tag{4}$$

(here $w_{n+1,n}$ is the vertical velocity at the $n+1/n$th interface, $\nabla^2\psi_n$ is the relative vorticity, βy is the planetary vorticity and f_0 is the value of the Coriolis parameter at the southern boundary),

and 2) the equations for the movement of the interfaces between the layers

$$w_{n+1,n} = W_{n+1,n} + \frac{D}{Dt}\eta_{n+1,n} \tag{5}$$

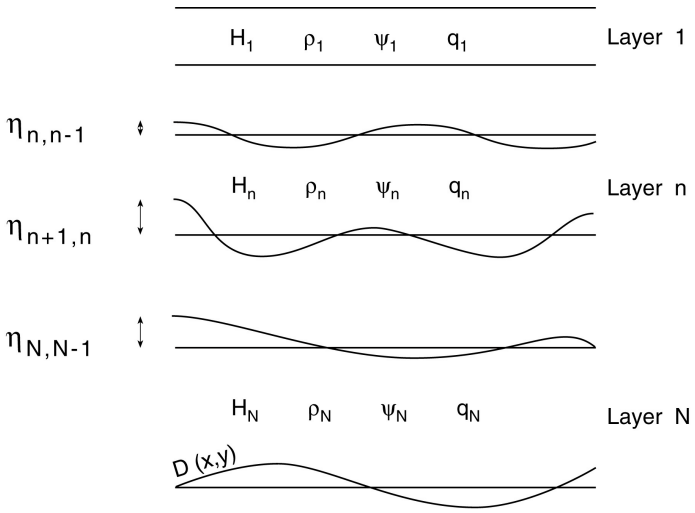

Figure 2.2: Schematic representation of the vertical model structure. (See text for meaning of symbols.)

where $\eta_{n+1,n}$ is the deviation of the interface between layers $(n+1)$ and n. Thus $D\eta/Dt$ represents the vertical velocity of the interface and the cross-interface velocities $W_{n+1,n}$ represent the transformation of fluid from one density to another, and may be considered as arising from heating terms in the temperature perturbation equation to which, applied at the thermodynamic levels, Eqn. (5) is equivalent.

The thermal wind relation gives the interface displacements in terms of the streamfunctions in the layers above and below thus

$$\eta_{n+1,n} = \frac{f_0\rho_0}{g\Delta\rho_{n+1,n}}(\psi_{n+1} - \psi_n) \tag{6}$$

where g is the gravitational acceleration and ρ_0 is the mean density of water.

Substitution of Eqn. (6) into Eqn.(5) gives (in fact proportional to) the 'buoyancy equation'

$$\frac{D}{Dt}(\psi_n - \psi_{n+1}) + \frac{g\Delta\rho_{n+1,n}}{\rho_0 f_0}(w_{n+1,n} - W_{n+1,n}) = 0 \tag{7}$$

or, expanding the substantial derivative

$$\frac{\partial}{\partial t}(\psi_n - \psi_{n+1}) + J(\psi_n, \psi_n - \psi_{n+1}) + \frac{g\Delta\rho_{n+1,n}}{\rho_0 f_0}(w_{n+1,n} - W_{n+1,n}) = 0. \tag{8}$$

Elimination of vertical velocities in Eqn. (4) with Eqn.(8) gives the Q-G potential vorticity equation

$$\frac{\partial q_n}{\partial t} + J(\psi_n, q_n) = G_n \tag{9}$$

where

$$\left.\begin{array}{rcl} q_1 & = & \nabla^2\psi_1 + \beta y - \frac{f_0^2\rho_0}{gH_1\Delta\rho_{21}}(\psi_1 - \psi_2) \\ q_n & = & \nabla^2\psi_n + \beta y - \left\{\frac{f_0^2\rho_0}{gH_n\Delta\rho_{n+1,n}}(\psi_n - \psi_{n+1}) + \frac{f_0^2\rho_0}{gH_n\Delta\rho_{n,n-1}}(\psi_n - \psi_{n-1})\right\} \\ q_N & = & \nabla^2\psi_N + \beta y - \frac{f_0^2\rho_0}{gH_N\Delta\rho_{N,N-1}}(\psi_N - \psi_{N-1}) + \frac{f_0 D}{H_N} \end{array}\right\} \tag{10}$$

are the quasi-geostrophic potential vorticities in each layer and

$$\left.\begin{array}{rcl} G_1 & = & \frac{f_0}{H_1}(w_0 - W_{21}) - \nu\nabla^6\psi_1 \\ G_n & = & \frac{f_0}{H_n}(W_{n,n-1} - W_{n+1,n}) - \nu\nabla^6\psi_n \\ G_N & = & \frac{f_0}{H_N}W_{N,N-1} - \nu\nabla^6\psi_N - \varepsilon\nabla^2\psi_N \end{array}\right\} \tag{11}$$

are the potential vorticity forcing functions.

Note 1) w_0 is the Ekman suction velocity imposed at the top of the surface layer and can be related to the imposed wind stress ($\underline{\tau}$) curl thus:

$$w_0 = \underline{k}.curl(\underline{\tau}/\rho_0 f). \tag{12}$$

2) The vertical velocity at the bottom of the ocean is given by

$$w_{bottom} = w_F + \frac{D}{Dt}D \tag{13}$$

where $D = D(x, y)$ is the (assumed small) bottom topography and w_F is the rate of ejection of fluid from the bottom Ekman layer. This has been parameterised in Eqn. (11) by assuming

$$w_F = \frac{\varepsilon H_N}{f_0}\nabla^2\psi_N. \tag{14}$$

3) It may be shown that the (equivalent) perturbation temperature $T_{n+1,n}$ at the interface is given by

$$T_{n+1,n} = \frac{f_0}{g\alpha}\frac{(\psi_n - \psi_{n+1})}{0.5(H_n + H_{n+1})} \tag{15}$$

where $\alpha = (-\partial\rho/\partial T)/\rho_0$ is the coefficient of thermal expansion. The cross-interface velocity between the nth and $(n+1)$th layers may then be related to the total heat input/unit area $\mathcal{H}_a$ experienced between the middle of the $(n+1)$th and nth layers by

$$W_{n+1,n} = \frac{\alpha}{C_p\Delta\rho_{n+1,n}}\mathcal{H}_a. \tag{16}$$

Values of $\alpha = 2 \times 10^{-4}$ K^{-1}, $C_p = 4000$ Jkg^{-1} K^{-1} appropriate to near surface waters gives (in ms^{-1})

$$W_{n+1,n} = 5 \times 10_a^{-8}\mathcal{H}_a/\Delta\rho_{n+1,n} \tag{17}$$

if $\Delta\rho_{n+1,n}$ is specified in kgm^{-3} and $\mathcal{H}_a$ in Wm^{-2}.

2.2 The non-dimensional equations

We choose to scale x and y by L_y, and H_n and D by H, the total depth of the ocean. We write the depth of the layer as a fraction of the total depth $\delta_n = H_n/H$, $d = D/H$, and the density jumps $\Delta\rho_{n+1,n}/\rho_0 = \Delta\sigma_{n+1,n} \times 10^{-3}$, where $\Delta\sigma_{n+1,n}$ is the difference in density expressed in σ units. If time is scaled by $(\beta L_y)^{-1}$ then for $L_y = 3000$ km and $\beta = 2 \times 10^{-11}$, this timescale $(\beta L_y)^{-1} \sim 10$ hours.

We scale ψ by $U_c L_y$ where U_c is a characteristic velocity, and scale q by βL_y. Then Eqn. (9) becomes

$$\frac{1}{R}\frac{\partial q_n}{\partial t} + J(\psi_n, q_n) = G'_n \tag{18}$$

where $R = U_c/(\beta L_y^2)$ is a Rossby number for the vorticity equation,

$$\left.\begin{array}{rcl} q_1 &=& R\nabla^2\psi_1 + y - \frac{RL_y^2}{\delta_1 L_T^2}\left\{\frac{\psi_1-\psi_2}{\Delta\sigma_{21}}\right\} \\ q_n &=& R\nabla^2\psi_n + y - \frac{RL_y^2}{\delta_n L_T^2}\left\{\frac{\psi_n-\psi_{n-1}}{\Delta\sigma_{n,n-1}} + \frac{\psi_n-\psi_{n+1}}{\Delta\sigma_{n+1,n}}\right\} \\ q_N &=& R\nabla^2\psi_N + y - \frac{RL_y^2}{\delta_N L_T^2}\left\{\frac{\psi_N-\psi_{N-1}}{\Delta\sigma_{N,N-1}}\right\} + \frac{f_0 d}{\beta L_y \delta_N} \end{array}\right\} \tag{19}$$

the 'pseudo Rossby radius' $L_T = \sqrt{\frac{gH}{10^3 f_0^2}}$ (of magnitude 70 km for $H = 5000$ m and $f_0 = 10^{-4}$ s^{-1}). The forcing terms (scaled by βU_c) are

$$\left.\begin{array}{rcl} G'_1 &=& \frac{1}{\delta_1}\frac{U_S}{U_c}(w_0 - W_{21}) - \frac{\nu}{\beta L_y^5}\nabla^6\psi_1 \\ G'_n &=& \frac{n}{\delta_n}\frac{U_S}{U_c}(W_{n,n-1} - W_{n+1,n}) - \frac{\nu}{\beta L_y^5}\nabla^6\psi_n \\ G'_N &=& \frac{N}{\delta_N}\frac{U_S}{U_c}W_{N,N-1} - \frac{\nu}{\beta L_y^5}\nabla^6\psi_N - \frac{\varepsilon}{\beta L_y}\nabla^2\psi_N \end{array}\right\} \tag{20}$$

where the Sverdrup velocity is given by

$$U_S = f_0 w_0/(\beta H). \tag{21}$$

After scaling, the thermodynamic equation becomes

$$\frac{\partial}{\partial t}(\psi_n - \psi_{n+1}) + RJ(\psi_n, \psi_n - \psi_{n+1}) + \frac{U_S L_T^2}{U_c L^2}\Delta\sigma_{n+1,n}(w_{n+1,n} - W_{n+1,n}) = 0. \tag{22}$$

We have scaled the vertical velocities W by w_0, a typical value of the Ekman pumping chosen to be 30 m yr^{-1} (10^{-6} ms^{-1}). U_S is the depth-averaged value of the meridional velocity implied by the Sverdrup balance resulting from this Ekman pumping.

In oceanographic studies we have taken the characteristic velocity $U_c = U_S$; for the values of β, f_0, H and w_0 chosen above, $U_S \sim 1$mms^{-1}.

2.3 Numerical integration of the quasi-geostrophic potential vorticity equation

The major steps in the numerical integration are now outlined.

A centred difference leapfrog scheme is used for the forward time-stepping of the finite difference version of the potential vorticity equation (18). Thus

$$q_{i,j}^{t+1} - q_{i,j}^{t-1} = 2\Delta t R \left\{ J_{i,j}^{t} + G_{i,j}^{\prime t-1} \right\} \tag{23}$$

where the $_{i,j}$ suffix denotes the value at the i,jth gridpoint, the $J_{i,j}^{t}$ is the Jacobian and the $q_{i,j}^{t-1}$ are the q's from the preceding timestep.

The Jacobian $J_{i,j}^{t}$ is calculated using values of q and ψ from the current timestep. The Arakawa Jacobian (Arakawa, 1966) formulation is used. This ensures that there is no net advection out of the region of interior points over which time-stepping is performed of 1) potential vorticity 2) enstrophy $\frac{1}{2}q^2$ or 3) energy, so long as ψ is constant along the boundary

$$\text{i.e.} \sum_{\substack{interior \\ points}} J_{i,j} = \sum_{\substack{interior \\ points}} q_{i,j} J_{i,j} = \sum_{\substack{interior \\ points}} \psi_{i,j} J_{i,j} = 0. \tag{24}$$

This stepping forward is only performed at the interior gridpoints. Values of ψ are then found everywhere by solving an elliptic equation with appropriate boundary conditions. The inversion of the Helmholtz equations does not require knowledge of q along the boundary. The new q along the boundary is, however, required in the calculation of the Jacobian at the points immediately adjacent to the boundary. It is determined by imposing the condition that the relative vorticity $\nabla^2\psi = 0$ on the boundaries.

Values of q and ψ from the previous timestep are used to calculate the frictional contributions (Ekman — $\nabla^2\psi_N$, and biharmonic — $\nabla^6\psi_n$) to the forcing terms G_n'. Calculation of $\nabla^2\psi$ is straightforward: the biharmonic friction is evaluated by repeated application of the finite difference ∇^2 operator to the above determined $\nabla^2\psi$ using the condition that $\nabla^4\psi = 0$ on the boundary.

The mechanical forcing (appearing as an Ekman suction velocity generated by the wind stress curl) and thermal forcing (imposed as interfacial velocities) are then calculated and added to the frictional terms to give the G_n''s.

2.4 Inverting the potential vorticity

Eqn. (19) may be written in matrix form, eg.

$$\underline{q} = \beta \underline{y} + \nabla^2 \underline{\psi} - \underline{A}\underline{\psi} \tag{25}$$

where $\underline{A}$ is a positive definite tridiagonal stretching matrix, not necessarily orthogonal, and $\underline{q}$, $\underline{y}$ and $\underline{\psi}$ are column vectors comprised of the values of q, y and ψ on the layers. The matrix $\underline{A}$ is defined by

$$\left.\begin{array}{ll}
A_{1,1} = F_T/\delta_1(\sigma_2 - \sigma_1) & \\
A_{1,2} = -A_{1,1} & \\
A_{i,i-1} = -F_T/\delta_i(\sigma_i - \sigma_{i-1}) & \\
A_{i,i} = -A_{i,i-1} - A_{i,i+1} & \left.\begin{array}{l} \\ \\ \end{array}\right\} i = 2, \ldots N-1 \\
A_{i,i+1} = -F_T/\delta_i(\sigma_{i+1} - \sigma_i) & \\
A_{N,N-1} = -F_T/\delta_N(\sigma_N - \sigma_{N-1}) & \\
A_{N,N} = -A_{N,N-1} & \\
A_{i,j} = 0 \qquad\qquad |i-j| \geq 2 &
\end{array}\right\} \tag{26}$$

where

$$F_T = \frac{10^3 U_c f_0^2}{\beta g H} \tag{27}$$

and H is the total model depth, U_c is the characteristic velocity and f_0 is the value of the Coriolis parameter at the southern boundary.

To solve for $\underline{\psi}$, we require that (i) $\underline{A}$ is non-singular [i.e. $det\underline{A} \neq 0 (\exists\, \underline{A}\, \mathrm{st}\, \underline{A}\,\underline{A}^{-1} = 1)$] and (ii) all eigenvalues of $\underline{A}$ are real. There are then N (not necessarily different) non-zero eigenvalues λ_m, with corresponding eigenvectors $\underline{e}^m$ such that

$$\underline{A}\,\underline{e}^m = \lambda_m \underline{e}^m, \tag{28}$$

where $\underline{e}^m = (e_1^m, e_2^m, \ldots e_N^m)^T$ (a column vector).

So if a function $\underline{g} = (g_1, g_2, \ldots, g_N)$, can be written

$$\underline{g} = g\underline{e}^m \tag{29}$$

(i.e. its values in different layers are in the same proportion as those of the eigenvectors $\underline{e}^m$), then

$$\nabla^2 \underline{g} - \underline{A}\,\underline{g} = \nabla^2 \underline{g} - \lambda_m \underline{g}. \tag{30}$$

We will use this to find a solution of Eqn. (25).

Because $\underline{A}$ is non-singular, these $\underline{e}^m$ are 'complete', i.e. we can express any

$$\left.\begin{array}{lcll} \underline{\psi} & = & \sum_{m=1}^N \psi^m \underline{e}^m & \\ \underline{y} & = & y\underline{1} & = \; y \sum_{m=1}^N 1^m \underline{e}^m \\ \underline{q} & = & \sum_{m=1}^N q^m \underline{e}^m & \end{array}\right\} \tag{31}$$

So, expressing Eqn. (25) in terms of the eigenvectors $\underline{e}^m$

$$q^m = \beta y 1^m + \nabla^2 \psi^m - \lambda_m \psi^m, \;\; m = 1, \ldots, N. \tag{32}$$

This results in N independent linear elliptic equations, which are readily solved (using a Helmholtz-equation solver) for ψ^m. It is then easy to recover $\psi = (\psi_1, \psi_2, \ldots, \psi_N)$ from the ψ^m by using Eqn. (31).

The method of solution is thus

(i) project q on to the eigenvectors $\underline{e}^m$, and find q^m and $\beta y 1^m$,

(ii) use a Helmholtz equation solver to find ψ^m,

(iii) recover ψ from ψ^m.

Consider Eqn. (25); suppose we pre-multiply by $\underline{E}^{-1}$, where

$$\underline{E} = (\underline{e}^1, \underline{e}^2, \ldots, \underline{e}^N) = \begin{pmatrix} e_1^1 & e_1^2 & \ldots & e_1^N \\ e_2^1 & e_2^2 & \ldots & e_2^N \\ \vdots & \vdots & & \vdots \\ e_N^1 & e_N^2 & \ldots & e_N^N \end{pmatrix}, \tag{33}$$

to give

$$\underline{\underline{E}}^{-1}\underline{q} = \beta y \underline{\underline{E}}^{-1}\underline{1} + \nabla^2 \underline{\underline{E}}^{-1}\underline{\psi} - \underline{\underline{E}}^{-1}\underline{\underline{A}}\,\underline{\underline{E}}\,\underline{\underline{E}}^{-1}\underline{\psi} \tag{34}$$

which can be written as

$$\underline{Q} = \beta y \underline{\Upsilon} + \nabla^2 \underline{\Psi} - \underline{\underline{D}}\,\underline{\Psi}, \tag{35}$$

where $\underline{Q} = \underline{\underline{E}}^{-1}\underline{q}$, $\underline{\Upsilon} = \underline{\underline{E}}^{-1}\underline{1}$, $\underline{\Psi} = \underline{\underline{E}}^{-1}\underline{\psi}$ are the column vectors composed of the projections of $\underline{q}$, $\underline{1}$ and $\underline{\psi}$ respectively on to the N eigenvectors $\underline{e}^m$, and

$$\underline{\underline{D}} = \underline{\underline{E}}^{-1}\underline{\underline{A}}\,\underline{\underline{E}}. \tag{36}$$

Now $\underline{\underline{E}}^{-1}\underline{\underline{E}} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$ and $\underline{\underline{E}} = (\underline{e}^1, \underline{e}^2, \dots, \underline{e}^N)$: the columns are eigenvectors.

From Eqn. (33) we can express

$$\underline{\underline{E}}^{-1} = \begin{pmatrix} \cdots & \underline{e}^{*1} & \cdots \\ \cdots & \underline{e}^{*2} & \cdots \\ & \vdots & \\ \cdots & \underline{e}^{*N} & \cdots \end{pmatrix} \tag{37}$$

where the rows are $\underline{e}^{*m}$, with

$$\underline{e}^{*m}.\underline{e}^n = \delta_{m,n}. \tag{38}$$

These $\underline{e}^{*m}$ are known as the complementary eigenvectors. Now consider $\underline{\underline{D}} = \underline{\underline{E}}^{-1}\underline{\underline{A}}\,\underline{\underline{E}}$:

$$\underline{\underline{A}}\,\underline{\underline{E}} = \underline{\underline{A}}(\underline{e}^1, \underline{e}^2, \dots, \underline{e}^N) = (\lambda_1\underline{e}^1, \lambda_2\underline{e}^2, \dots, \lambda_N\underline{e}^N). \tag{39}$$

Hence

$$\underline{\underline{D}} = \begin{pmatrix} \cdots & \underline{e}^{*1} & \cdots \\ \cdots & \underline{e}^{*2} & \cdots \\ & \vdots & \\ \cdots & \underline{e}^{*N} & \cdots \end{pmatrix} (\lambda_1\underline{e}^1, \lambda_2\underline{e}^2, \dots, \lambda_N\underline{e}^N) = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_N \end{pmatrix}. \tag{40}$$

$\underline{\underline{D}}$ is a diagonal matrix composed of eigenvalues, and $\underline{\psi}$ (the values of ψ on each of the layers) may be found from the modal projections $\underline{\Psi}$ using

$$\underline{\psi} = \underline{\underline{E}}\,\underline{\Psi} = (\underline{e}^1, \underline{e}^2, \dots, \underline{e}^N) \begin{pmatrix} \psi^1 \\ \psi^2 \\ \cdots \\ \psi^N \end{pmatrix}. \tag{41}$$

2.4.1 Boundary conditions

We now consider the boundary conditions to be imposed on the side boundary values of ψ^m when evaluating them from the q^m [by inversion of the Helmholtz equation (32)]. For the barotropic mode, it is clearly acceptable to require $\psi^1 = 0$ along the side boundaries. However, it is less clear what are the appropriate boundary conditions on the baroclinic modes. By considering the horizontally-averaged mean interface displacements, conservation of mass requires

$$\frac{\partial}{\partial t}\left\{\overline{\psi_{n+1} - \psi_n}^{xy}\right\} = \frac{g\Delta\rho_{n+1,n}}{\rho_0 f_0}\left(\overline{w_{n+1,n} - W_{n+1,n}}^{xy}\right) \tag{42}$$

where $\overline{\phi}^{xy}$ denotes a horizontal average of ϕ over the basin, $w_{n+1,n}$ is the vertical velocity at the interface and $W_{n+1,n}$ is the cross-interface velocity resulting from buoyancy forcing. Note that in the horizontal averaging, half weight must be given to points on the boundary and quarter weight to corner points, since the condition $\psi = 0$ is applied at the gridpoints. The horizontal advection terms must sum to zero since there can be no flow through the side boundaries.

Initially, we will require $w_{n+1,n} = W_{n+1,n} = 0$, which represents the condition of no net vertical motion or buoyancy forcing. Since the mean interfacial displacements are proportional to $\overline{\Xi_{n+1,n}}^{xy} = \overline{\psi_{n+1} - \psi_n}^{xy}$, it is clear that these $(N-1)$ $\overline{\Xi_{n+1,n}}^{xy}$ (there are N layers) are determined if we know the mean vertical velocities and buoyancy forcings.

Let

$$\underline{X} = \left(\overline{\psi^1}^{xy}, \overline{\Xi_{21}}^{xy}, \ldots, \overline{\Xi_{N,N-1}}^{xy}\right)^T = \begin{pmatrix} \sum_{n=1}^{N} \underline{e}^{*1}\overline{\psi_n}^{xy} \\ \overline{\psi_2 - \psi_1}^{xy} \\ \vdots \\ \overline{\psi_N - \psi_{N-1}}^{xy} \end{pmatrix}$$

$$= \begin{pmatrix} & & \underline{e}^{*1} & & \\ -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \begin{pmatrix} \overline{\psi_1}^{xy} \\ \overline{\psi_2}^{xy} \\ \overline{\psi_3}^{xy} \\ \vdots \\ \overline{\psi_N}^{xy} \end{pmatrix} = \underline{P}\,\overline{\underline{\psi}}^{xy}.$$

As long as $\underline{P}$ is non-singular ($\det\underline{P} \neq 0$), we can write

$$\overline{\underline{\psi}}^{xy} = \underline{P}^{-1}\underline{X}. \tag{43}$$

But we can express the modal projection in terms of levels thus:

$$\underline{\Psi} = \underline{E}^{-1}\underline{\psi}, \tag{44}$$

so the mean modal projections $\overline{\underline{\Psi}}^{xy}$ must satisfy

$$\overline{\underline{\Psi}}^{xy} = \underline{E}^{-1}\underline{P}^{-1}\underline{X} = \underline{E}_2\underline{X} = \underline{\chi}. \tag{45}$$

Thus the continuity equation (42) has given us the χ^m, the horizontally-averaged values of the baroclinic modes. These will generally be different from the $\overline{\psi_0^m}^{xy}$, the spatial averages of the solutions derived by inverting

$$\nabla^2 \psi_0^m - \lambda^m \psi_0^m = q^m \tag{46}$$

with $\psi_0^m = 0$ on the boundaries.

The solution is thus found by adding a multiple c of the homogeneous solution ψ_b^m which obeys

$$\nabla^2 \psi_b^m - \lambda_m \psi_b^m = 0 \tag{47}$$

with $\psi_b^m = 1$ on the boundaries, such that $\overline{\psi_0^m + c\psi_b^m}^{xy} = \chi^m$.

Thus (i) Eqn. (46) is used to calculate ψ_0^m, (ii) c is set equal to $\left(\chi^m - \overline{\psi_0^m}^{xy}\right) / \overline{\psi_b^m}^{xy}$ (the ψ_b^m fields do not change and therefore need only be calculated once), and (iii) $c\psi_b^m$ is added to the ψ_0^m fields to give the new ψ^m field.

2.4.2 Summary

In the model the following procedure is adopted:

(1) We define the matrix $\underline{A}$.

(2) The eigenvalues λ_m of $\underline{A}$ and the matrix $\underline{E}$ are found; the column vectors of $\underline{E}$ are the eigenvectors of $\underline{A}$, defined by

$$\underline{A}\,\underline{e}^m = \lambda_m \underline{e}^m. \tag{48}$$

(3) The inverse matrix of $\underline{E}$, $\underline{E}^{-1}$, is computed.

(4) The matrix product of $\underline{P}$ and $\underline{E}$ is calculated by multiplying the two matrices

$$\underline{P} = \begin{pmatrix} e_1^{*1} & e_2^{*1} & \dots & e_N^{*1} \\ -1 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \underline{E} = \begin{pmatrix} e_1^1 & e_1^2 & \dots & e_1^N \\ e_2^1 & e_2^2 & \dots & e_2^N \\ \vdots & \vdots & & \vdots \\ e_N^1 & e_N^2 & \dots & e_N^N \end{pmatrix}. \tag{49}$$

(5) The inverse of this product is evaluated to give $\underline{E}_2$

$$\underline{E}_2 = (\underline{P}\,\underline{E})^{-1} = \underline{E}^{-1}\underline{P}^{-1}. \tag{50}$$

(6) All the elements of $\overline{\Xi}^{xy}$ are set to zero initially

(7) ψ_b^m is set to 1 on the boundaries for modes 2 to N. The interior values are then calculated (these remain fixed throughout the simulation) and $\overline{\psi_b^m}^{xy}$ is found by horizontal averaging of ψ_b^m. The steps which are taken in inverting the q field to give the ψ field each timestep now follow:

(8) $\overline{\Xi}^{xy}$ is found for levels 2 to N by determining the mean interfacial displacements $\left(\overline{\psi_i - \psi_{i-1}}^{xy}\right)$.

(9) ψ_0^m is set to zero on all the boundaries.

(10) After subtracting βy and the topography from q, q^m is found by projecting q on to the modes, eg.

$$q^m = \sum_{k=1}^{N} (E_k^{-1})^m q_k \tag{51}$$

where k is the level number, and m is the mode number.

(11) The Helmholtz equation solver is used to invert the q^m to give the ψ_0^m for all the internal gridpoints $\overline{\psi_0^m}^{xy}$ is then calculated by horizontal averaging.

(12) For the first (barotropic) mode,$\overline{\Xi}^{xy}$ is equated to $\overline{\psi_0^1}^{xy}$ (i.e. the first element in vector $\underline{X}$) while for the other (baroclinic) modes χ^m is found by evaluating the product of $\underline{E_2}$ and $\underline{X}$. The coefficient c is then found using

$$c = (\chi^m - \overline{\psi_0^m}^{xy})/\overline{\psi_b^m}^{xy}. \tag{52}$$

(13) ψ^m is then evaluated using $\psi_0^m + c\psi_b^m$.

(14) The streamfunction in each layer of the model is then determined from the modes using Eqn. (31).

2.5 Useful references

A. Arakawa (1966): Computational Design for Long-Term Numerical Integrations of the Equations of Atmospheric Motion. *Journal of Computational Physics* **1** 119-143.
D.R. Moore (1986): Efficient real FFTs for rapid elliptic and spectral codes. *Journal of Computational Physics.*