**Q1.** *Using Taylor series, derive a second order finite-difference approximation to $\partial_x f$ at the location $x_i$ using a three point stencil involving $f_{i-1}$, $f_i$ and $f_{i+1}$.*

Taylor series for $f_{i\pm1}$ at $x = x_i$ are

$$f_{i\pm1} = f_i \pm \Delta x_{i\pm\frac{1}{2}} f'(x_i) + \frac{\Delta x_{i\pm\frac{1}{2}}}{2} f''(x_i) \pm \frac{\Delta x^2_{i\pm\frac{1}{2}}}{3!} f'''(x_i) + \frac{\Delta x^3_{i\pm\frac{1}{2}}}{4!} f''''(x_i) + H.O.T.$$

We pose that $f'(x_i)$ can be expressed as

$$f'(x_i) = a f_{i+1} + b f_i + c f_{i-1} + O(\Delta x^2)$$

and substitute $f_{i\pm1}$ with their Taylor series. Collecting like terms, we see that

$$\begin{aligned} a + b + c &= 0, \\ a\frac{\Delta x_{i+\frac{1}{2}}}{2} - c\frac{\Delta x_{i-\frac{1}{2}}}{2} &= 1, \\ a\frac{\Delta x^2_{i+\frac{1}{2}}}{6} + c\frac{\Delta x^2_{i-\frac{1}{2}}}{6} &= 0. \end{aligned}$$

The solution to the last two equations is

$$\begin{aligned} a &= \left(\frac{1}{\Delta_{i-\frac{1}{2}} + \Delta_{i+\frac{1}{2}}}\right)\frac{\Delta_{i-\frac{1}{2}}}{\Delta_{i+\frac{1}{2}}} = \frac{\Delta^2_{i-\frac{1}{2}}}{(\Delta_{i-\frac{1}{2}} + \Delta_{i+\frac{1}{2}})\Delta_{i+\frac{1}{2}}\Delta_{i-\frac{1}{2}}} \\ c &= \left(\frac{-1}{\Delta_{i-\frac{1}{2}} + \Delta_{i+\frac{1}{2}}}\right)\frac{\Delta_{i+\frac{1}{2}}}{\Delta_{i-\frac{1}{2}}} = \frac{-\Delta^2_{i+\frac{1}{2}}}{(\Delta_{i-\frac{1}{2}} + \Delta_{i+\frac{1}{2}})\Delta_{i+\frac{1}{2}}\Delta_{i-\frac{1}{2}}} \end{aligned}$$

and using the first equation, $b = -a - c$, gives

$$b = \frac{\Delta^2_{i+\frac{1}{2}} - \Delta^2_{i-\frac{1}{2}}}{(\Delta_{i-\frac{1}{2}} + \Delta_{i+\frac{1}{2}})\Delta_{i+\frac{1}{2}}\Delta_{i-\frac{1}{2}}}.$$

The full expression for $\partial_x f(x_i)$ is

$$f'(x_i) \approx \frac{\Delta^2_{i-\frac{1}{2}} f_{i+1} + (\Delta^2_{i+\frac{1}{2}} - \Delta^2_{i-\frac{1}{2}}) f_i - \Delta^2_{i+\frac{1}{2}} f_{i-1}}{(\Delta_{i-\frac{1}{2}} + \Delta_{i+\frac{1}{2}})\Delta_{i+\frac{1}{2}}\Delta_{i-\frac{1}{2}}}.$$

Note that this can be written

$$f'(x_i) \approx \left(\frac{\Delta_{i-\frac{1}{2}}}{\Delta_{i-\frac{1}{2}} + \Delta_{i+\frac{1}{2}}}\right)\left(\frac{f_{i+1} - f_i}{\Delta_{i+\frac{1}{2}}}\right) + \left(\frac{\Delta_{i+\frac{1}{2}}}{\Delta_{i-\frac{1}{2}} + \Delta_{i+\frac{1}{2}}}\right)\left(\frac{f_i - f_{i-1}}{\Delta_{i-\frac{1}{2}}}\right).$$

**Q2.a** *i) Derive the actual truncation error terms out to order $\Delta x^2$.*

$$\frac{f_{i+1} - f_i}{\Delta x} = f'(x_i) + \frac{1}{2}\Delta x f''(x_i) + \frac{1}{6}\Delta x^2 f'''(x_i) + O(\Delta x^3)$$

*ii) Derive an approximation for $\partial_{xx} f$ evaluated at the same point, $x_i$, that uses the values $f_{i-1}$, $f_i$ and $f_{i+1}$.*

$$f''(x_i) = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2} + O(\Delta x^2 f'''')$$

*iii) Substitute your approximation to $\partial_{xx} f$ from (ii) into the leading truncation term. What is the leading order truncation term and resulting scheme?*

$$\frac{f_{i+1} - f_i}{\Delta x} = f'(x_i) + \frac{1}{2}\Delta x \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2} + \frac{1}{6}\Delta x^2 f'''(x_i) + O(\Delta x^3)$$

or, moving the terms in $f_{i-1}$, $f_i$ and $f_{i+1}$ to the L.H.S.,

$$\frac{f_{i+1} - f_{i-1}}{2\Delta x} = f'(x_i) + \frac{1}{6}\Delta x^2 f'''(x_i) + O(\Delta x^4)$$

The largest remaining truncation term is $O(\Delta x^2)$ and so this is a centered second order difference approximation to $\partial_x f$ at $x = x_i$ (i.e. $f'(x_i)$).

**Q2.b** *i) Derive the leading order truncation error.*

As above:

$$\frac{f_{i+1} - f_{i-1}}{2\Delta x} = f'(x_i) + \frac{1}{6}\Delta x^2 f'''(x_i) + O(\Delta x^3)$$

*ii) Using the stencil ($f_{i-2}$, $f_{i-1}$, $f_i$, $f_{i+1}$) write a finite difference approximation for $\partial_{xxx} f$.*

$$f'''(x_i) = \frac{f_{i+1} - 3f_i + 3f_{i-1} - f_{i-2}}{\Delta x^3} + O(\Delta x)$$

*iii) Substitute your approximation from (ii) into the leading order error term of the approximation to $\partial_x f$. This is an approximation for $\partial_x f$ of what order?*

$$\frac{f_{i+1} - f_{i-1}}{2\Delta x} = f'(x_i) + \frac{1}{6}\Delta x^2 \frac{f_{i+1} - 3f_i + 3f_{i-1} - f_{i-2}}{\Delta x^3} + O(\Delta x^3)$$

or, moving the terms in $f_{i-2}$, $f_{i-1}$, $f_i$ and $f_{i+1}$ to the L.H.S.,

$$\frac{2f_{i+1} + 3f_i - 6f_{i-1} + f_{i-2}}{6\Delta x} = f'(x_i) + O(\Delta x^3)$$

This is a third order difference approximation to $f'(x_i)$.

**Q2.c** *Do you see the pattern behind the methods you used in Q2.a and b. Briefly explain, how you would derive an $O(\Delta x^n)$ approximation to a finite difference expression if you were given an $O(\Delta x^{n-1})$ finite difference expression.*

Replacing the leading Taylor series term in a finite difference approximation with a finite difference approximation of that term results in a higher order scheme.

**Q3.** *i) Derive a second order expression of a similar form as before but using only the values $f_{i-2}$ and $f_{i+2}$.*

Literally replacing $\Delta x$ with $2\Delta x$:

$$\frac{f_{i+2} - f_{i-2}}{4\Delta x} = f'(x_i) + \frac{4}{6}\Delta x^2 f'''(x_i) + O(\Delta x^4)$$

*ii) Linearly combine your approximations from (i) and the former approximation to yield a $O(\Delta x^4)$ approximation for $\partial_x f$ at $x_i$.*

$$\begin{aligned} a\left(\frac{f_{i+2} - f_{i-2}}{4\Delta x}\right) + b\left(\frac{f_{i+1} - f_{i-1}}{2\Delta x}\right) &= a\left(f'(x_i) + \frac{4}{6}\Delta x^2 f'''(x_i)\right) \\ &\quad + b\left(f'(x_i) + \frac{1}{6}\Delta x^2 f'''(x_i)\right) + O(\Delta x^4) \end{aligned}$$

To eliminate the $O(\Delta x^2)$ terms and obtain an expression for $f'(x_i)$ we must solve

$$\begin{aligned} a + b &= 1 \\ \frac{4}{6}a + \frac{1}{6}b &= 0 \end{aligned}$$

the solution to which is

$$a = \frac{-1}{3} \quad \text{and} \quad b = \frac{4}{3}.$$

The resulting $O(\Delta x^4)$ approximation to $f'(x_i)$ is

$$\frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12\Delta x} = f'(x_i) + O(\Delta x^4)$$

**Q4.** *i) Write down an $O(\Delta x^3)$ approximation for $F_{i+\frac{1}{2}}$ in terms of $\theta_{i-1}$, $\theta_i$, $\theta_{i+1}$.*

Taylor series for $\theta_{i-1}$, $\theta_i$ and $\theta_{i+1}$ about $x = x_{i+\frac{1}{2}}$ are:

$$\begin{aligned}
\theta_{i+1} &= \theta(x_{i+\frac{1}{2}}) + \frac{\Delta x}{2}\theta'(x_{i+\frac{1}{2}}) + \frac{\Delta x^2}{2^2.2!}\theta''(x_{i+\frac{1}{2}}) + O(\Delta x^3) \\
\theta_i &= \theta(x_{i+\frac{1}{2}}) - \frac{\Delta x}{2}\theta'(x_{i+\frac{1}{2}}) + \frac{\Delta x^2}{2^2.2!}\theta''(x_{i+\frac{1}{2}}) + O(\Delta x^3) \\
\theta_{i-1} &= \theta(x_{i+\frac{1}{2}}) - \frac{3\Delta x}{2}\theta'(x_{i+\frac{1}{2}}) + \frac{3^2\Delta x^2}{2^2.2!}\theta''(x_{i+\frac{1}{2}}) + O(\Delta x^3)
\end{aligned}$$

We want an $O(\Delta x^3)$ approximation for $\theta(x_{i+\frac{1}{2}})$ in the form

$$a\theta_{i+1} + b\theta_i + c\theta_{i-1} = \theta(x_{i+\frac{1}{2}}) + O(\Delta x^3)$$

so

$$\begin{aligned}
a + b + c &= 1 \\
a - b - 3c &= 0 \\
a + b + 9c &= 0
\end{aligned}$$

First + last equations give $8c = -1$. Last two give $2a = -6c$. Difference of last two give $2b = -12c$.

$$a = \frac{3}{8} \quad ; \quad b = \frac{6}{8} \quad ; \quad c = \frac{-1}{8}$$

Third order interpolation for $\theta(x_{i+\frac{1}{2}})$ gives the stencil

$$u\,[c \quad b \quad a] = u\left[\frac{-1}{8} \quad \frac{6}{8} \quad \frac{3}{8}\right]$$

for $F_{i+\frac{1}{2}}$.

*ii) Substitute your $O(\Delta x^3)$ expressions for $F_{i+\frac{1}{2}}$ and $F_{i-\frac{1}{2}}$ in and derive the leading order truncation error.*

We need Taylor series expanded about $x = x_i$:

$$\begin{aligned}
\theta_{i+1} &= \theta(x_i) + \Delta x\theta'(x_i) + \frac{\Delta x^2}{2!}\theta''(x_i) + \frac{\Delta x^3}{3!}\theta'''(x_i) + O(\Delta x^4) \\
\theta_{i-1} &= \theta(x_i) - \Delta x\theta'(x_i) + \frac{\Delta x^2}{2!}\theta''(x_i) - \frac{\Delta x^3}{3!}\theta'''(x_i) + O(\Delta x^4) \\
\theta_{i-2} &= \theta(x_i) - 2\Delta x\theta'(x_i) + \frac{2^2\Delta x^2}{2!}\theta''(x_i) - \frac{2^3\Delta x^3}{3!}\theta'''(x_i) + O(\Delta x^4)
\end{aligned}$$

$$\begin{aligned}
\partial_t\theta_i \quad + \quad & \frac{u}{\Delta x}\left[a\theta_{i+1} + (b-a)\theta_i + (c-b)\theta_{i-1} - c\theta_{i-2}\right] \\
= \partial_t\theta_i \quad & +(a + \quad (b-a) + (c-b) - \quad c)\frac{u\theta_i}{\Delta x} \\
& +(a + 0(b-a) - (c-b) + 2c)u\theta' \\
& +(a + 0(b-a) + (c-b) - 2^2c)\frac{u\Delta x}{2!}\theta'' \\
& +(a + 0(b-a) - (c-b) + 2^3c)\frac{u\Delta x^2}{3!}\theta''' + O(\Delta x^3) \\
= \partial_t\theta_i \quad & +0\theta_i + (a+b+c)u\theta' + (a-b-3c)u\Delta x\theta'' + (a+b+7c)u\Delta x^2\theta''' + O(\Delta x^3) \\
= \partial_t\theta_i \quad & +u\theta' + 2u\Delta x^2\theta''' + O(\Delta x^3)
\end{aligned}$$

So the $O(\Delta x^2)$ remains (since $a+b-7c = 10/8$) and the governing equation is only second order, not third order accurate, in $\Delta x$.

*We substituted a third order approximation of the flux so why is the approximation to the governing equation not third order?*

The difference approximation of $\partial_x F$

$$\partial_x F \approx \frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x}$$

is only second order accurate. No matter how accurate we make $F$, there will be a second order truncation error **unless** we can construct the truncation errors in $F_{i+\frac{1}{2}}$ and $F_{i-\frac{1}{2}}$ so as to cancel each other. As we do next...

*iii) Write down the $O(\Delta x^3)$ finite difference approximation to the governing equation evaluated at $x_i$ using the stencil ($\theta_{i-2}$, $\theta_{i-1}$, $\theta_i$, $\theta_{i+1}$).*

$$\partial_t\theta_i + \frac{u}{\Delta x}\left[\frac{2}{6}\theta_{i+1} + \frac{3}{6}\theta_i - \theta_{i-1} + \frac{1}{6}\theta_{i-2}\right]$$

*iii) Deduce an approximation for the flux $F_{i+\frac{1}{2}}$ that yields an $O(\Delta x^3)$ approximation to the linear advection problem.*

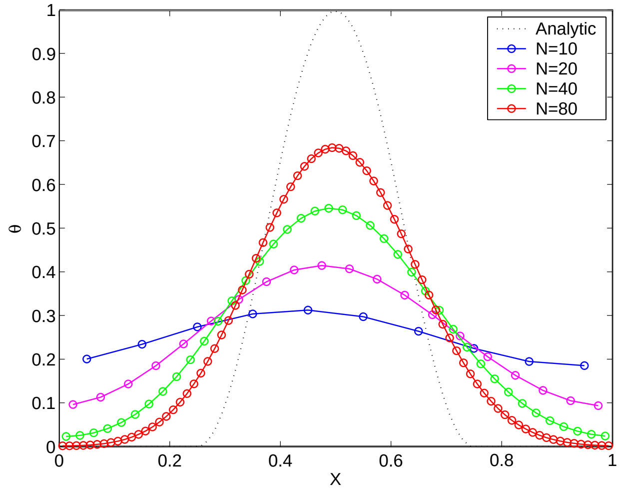
$$\begin{bmatrix}\frac{1}{6} & \frac{-6}{6} & \frac{3}{6} & \frac{2}{6}\end{bmatrix} = \begin{bmatrix}-\alpha & \alpha-\beta & \beta-\gamma & \gamma\end{bmatrix}$$

Solving for $\alpha$, $\beta$ and $\gamma$:

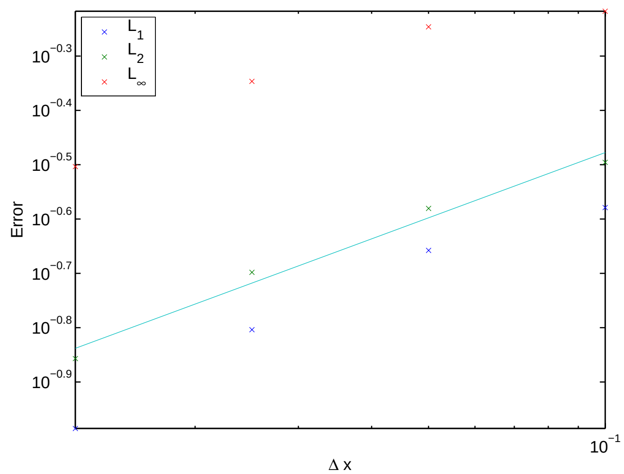
$$\alpha = \frac{-1}{6} \quad ; \quad \beta = \frac{5}{6} \quad ; \quad \gamma = \frac{2}{6}$$

**Q5** (MATLAB)

*i) Plot the initial conditions and solutions at time $t = 1$ for $N = 10, 20, 40, 80$.*



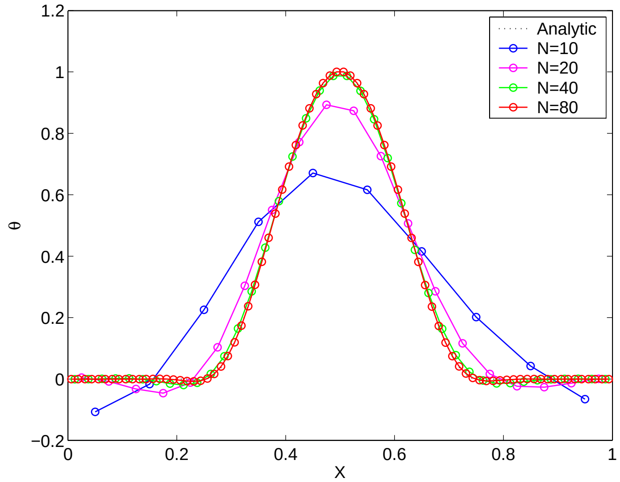

*ii) At each resolution, measure the $l_1$, $l_2$ and $l_\infty$ norms. Plot them as a function of grid-spacing and measure the power dependence of the $l_2$ curve on $\Delta x$.*

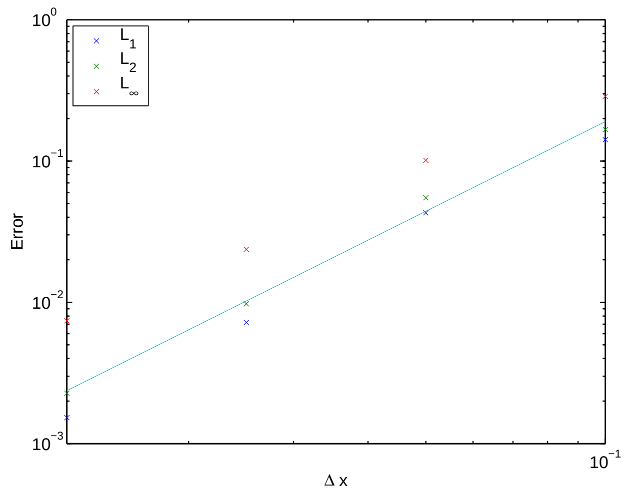



A fit to the $l_2$-norm points gives a power of $\Delta x^{0.4}$.

*iii) Repeat (i) and (ii) using the two forms of "third" order flux in Q4. Which is more accurate?*

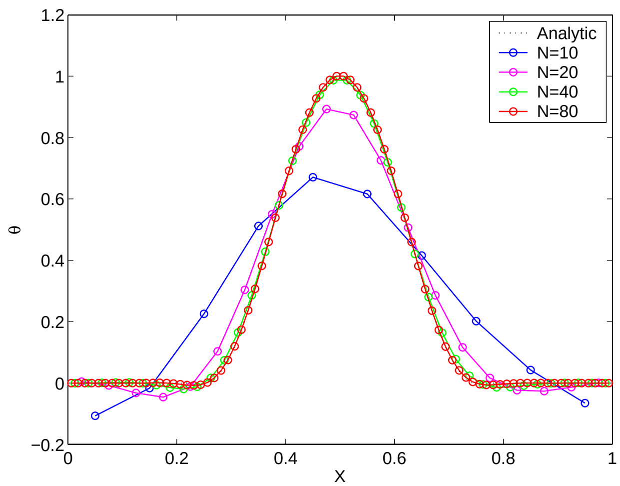
Third order flux (1/6 form)

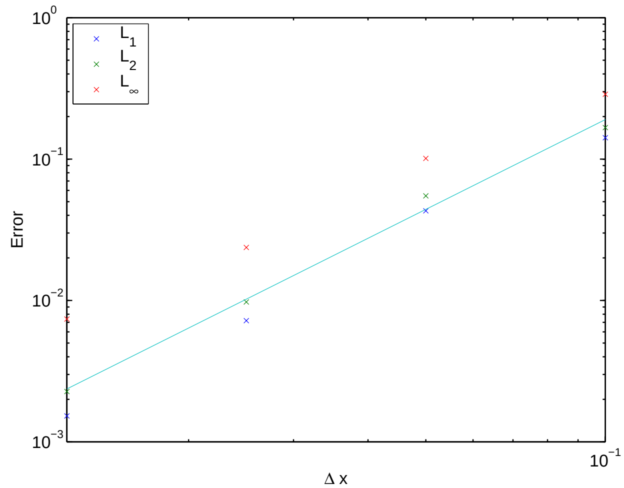



A fit to the $l_2$-norm points gives a power of $\Delta x^{2.1}$.

Third order flux (1/8 form)




A fit to the $l_2$-norm points gives a power of $\Delta x^{1.8}$. Using $\frac{1}{6}$ form is more accurate than the $\frac{1}{8}$ form, consistent with the analysis.

*iv) Why is the dependence not what you would expect, given all the effort you put into deriving the truncation errors in the previous questions?*

We are using the forward time-stepping scheme which is of order $\Delta t$. Although $\Delta t$ was reduced with $\Delta x$, the first order accuracy of the time differencing error dominates and higher order errors in the spatial differences.