9. Fourier Series as Least-Squares

Discrete Fourier series (6.29) are an exact representation of the sampled function if the number of basis functions (sines and cosines) are taken to equal the number, N, of data points. Suppose we use a number of terms $N' \leq N$, and seek a least-squares fit. That is, we would like to minimize

$$J = \sum_{t=0}^{T-1} \left(x_t - \sum_{m=1}^{[N'/2]} \alpha_m e^{2\pi imt/T} \right) \left(x_t - \sum_{m=1}^{[N'/2]} \alpha_m e^{2\pi imt/T} \right)^* . \tag{9.1}$$

Taking the partial derivatives of J with respect to the a_m and setting to zero (generating the least-squares normal equations), and invoking the orthogonality of the complex exponentials, one finds that (1) the governing equations are perfectly diagonal and, (2) the a_m are given by precisely (6.29, 6.30). Thus we can draw an important conclusion: *a Fourier series, whether partial or complete, represents a least-squares fit of the sines and cosines to a time series.* Least-squares is discussed at length in Wunsch (1996).

Exercise. Find the normal equations corresponding to (9.1) and show that the coefficient matrix is diagonal.

Non-Uniform Sampling

This result (9.1) shows us one way to handle a non-uniformly spaced time series. Let $x(t)$ be sampled at arbitrary times t_j. We can write

$$x(t_j) = \sum_{m=1}^{[N'/2]} \alpha_m e^{2\pi imt_j/T} + \varepsilon_j \tag{9.2}$$

where ε_j represents an error to be minimized as

$$J = \sum_{j=0}^{j_N-1} \varepsilon_j^2 = \sum_{j=0}^{j_N-1} \left(x(t_j) - \sum_{m=1}^{[N'/2]} \alpha_n e^{2\pi int_j/T} \right) \left(x(t_j) - \sum_{m=1}^{[N'/2]} \alpha_m e^{2\pi imt_j/T} \right)^* , \tag{9.3}$$

or the equivalent real form, and the normal equations found. The resulting coefficient matrix is no longer diagonal, and one must solve the normal equations by Gaussian elimination or other algorithm. If the record length and/or N' is not too large, this is a very effective procedure. For long records, the computation can become arduous. Fortunately, there exists a fast solution method for the normal equations, generally called the Lomb-Scargle algorithm (discussed, e.g., by Press et al., 1992; an application to intermittent satellite sampling of the earth's surface can be seen in Wunsch (1991)). The complexities of the algorithm should not however, mask the underlying idea, which is just least-squares.

Exercise. Generate a uniformly spaced time series, x_t, by specifying the coefficients of its Fourier series. Using the Fourier series, interpolate x_t to generate an irregularly spaced set of values of $x(t_j)$. Using $x(t_j)$ and the normal equations derived from (9.3), determine the Fourier components. Discuss their relationship to the known ones. Study what happens if the $x(t_j)$ are corrupted by the addition of white noise. What happens if the observation times t_j are corrupted by a white noise error? ("White noise" is defined below. For present purposes, it can be understood as the output of an ordinary pseudo-random number generator on your computer.)

Irregular Sampling Theorems

There are some interesting theoretical tools available for the discussion of infinite numbers of *irregularly* spaced perfect samples of band-limited functions. A good reference is Freeman (1965), who gives explicit expressions for reconstruction of band-limited functions from arbitrarily spaced data. Among the useful results are that any regularly spaced sample which is missing, can be replaced by an arbitrary irregularly spaced sample anywhere in the record. The limiting case of the expressions Freeman shows, would suggest that one could take *all* of the samples and squeeze them into an arbitrarily brief time interval. This inference would suggest a strong connection between band-limited functions and analytic functions describable by their Taylor Series (regarding closely spaced samples as being equivalent to first, second, etc. differences). Related results permit one to replace half the samples by samples of the derivatives of the function, etc. The reader is cautioned that these results apply to infinite numbers of perfect samples and their use with finite numbers of inaccurate data has to be examined carefully.

Some useful results about "jittered" sampling can be seen in Moore and Thomson (1991), and Thomson and Robinson (1996); an application to an ice core record is Wunsch (2000).