12.864 Inference from Data and Models 23 April 2003
Problem Set No. 4 Due: 2 May 2003

1. Consider the mass-spring oscillator

$$\frac{d^2\xi}{dt^2} + 2\xi(t) = q(t).$$

Discretize it in finite differences (your choice of how to do it) , and write it in the canonical statespace form for $\xi(n\Delta t), \Delta\ t = 0.2$. The discrete version of $q(t)$ is a simple zero-mean white-noise process of variance unity. An estimate of the intial conditions is $\xi\ (0) = 1,\ \xi\ (\Delta t) = 0$, where $\left\langle (\xi(0) - \xi(0))^2 \right\rangle = 1$; $\left\langle (\xi(1) - \xi(\Delta t))^2 \right\rangle = 2$. There are no observations until $t = 20\Delta t$, when $y(20\Delta t) = \xi(20\Delta t) + n(20\Delta t) = 8 \pm 1$.

a. Using a Kalman filter code of your own devising, make a best-estimate of $\xi(n\Delta t), 0 \leq n \leq 22$ along with an uncertainty estimate of your result.

b. Using the RTS smoother make an estimate of $q(n\Delta t)$ in this same interval and of the actual initial conditions.

2. Using the method of Lagrange multipliers (adjoint method), re-solve problem 1b. Do the answers differ?

3. (I won't look at this, but you need it for problem 4.) Using a numerical, discrete fourier transform code (your choice), calculate the Fourier transform (or series) of $x_t = \sin(2\pi t/16),\ 0 \leq t \leq 256$. Convince yourself that your understand the answer, and that by an inverse transform you can recover x_t. Do the same for $\cos(2\pi t/16)$, for $x_t = \delta_{t,0},\ x_t = \delta_{t,100}$. ($t$ is now a discrete integer).

4. Now generate a 512 point white noise time series η_t, using a pseudo-random number generator with known variance.

(a) Compute its Fourier series coefficients and plot a histogram of the real and imaginary (or in and out of phase) parts. Do the same for the magnitudes-squared and phases. Comment on what you see. Plot them all as a function of frequency. What is the relationship between the mean-square of the Fourier series coefficients and that of the η_t?

(b) Now form a new time series,

$$y_t = \eta_t + .9\eta_{t-1},$$

and repeat part (a). Comment on any difference.

(c) Let γ_n^2 be the magnitude-squared of the Fourier coefficients (or transform) of y_t as a function of frequency (labelled by n). Average the γ_n^2 in overlapping groups of 5. What can you deduce from this result? Can you determine the coefficients relating y_t to η_t?