

Chapter 3

A statistical description of turbulence

The evolution of turbulent flows is very complex. Turbulent flows appear highly disorganized with structure at all scales. Signals appear unpredictable in their detailed behavior. However some statistical properties of the flow can be quite reproducible. This suggests that it can be useful to seek a statistical description of turbulent flows. A statistical measure is an average of some kind: over the symmetry coordinates, if any are available (*e.g.*, a time average for stationary flows); over multiple realizations (*e.g.*, an ensemble); or over the phase space of solutions if the dynamics are homogeneous.

In lecture 2 we have shown that a similar behavior is observed in simple deterministic maps. These maps are chaotic and not predictable in their detailed properties, but their statistical properties are reproducible, just like for turbulent flows.

Thus it seems quite appropriate to introduce a probabilistic description of turbulence. However we know that the basic Boussinesq equations are deterministic. How can chance and chaos arise in a purely deterministic context? A nice discussion of this issue can be found, in chapter 3 of Frisch's book.

In this lecture we first introduce the statistical tools used in the analysis of turbulent flows. Then we show how to apply these tools to the study of turbulence.

3.1 Probability density functions and moments

A complete description of a turbulent variable v at a given location and instant in time is given by the **probability density function (PDF)**, $P(v)$, where $P(v)dv$ is

the probability of the variable v taking a value between v and $v + dv$, and

$$\int_{-\infty}^{+\infty} P(v)dv = 1. \quad (3.1)$$

The PDF can be estimated by constructing a histogram of an ensemble of measurements of v at the specified location, repeating the experiment time and time again. The larger the ensemble (i.e. the more times the experiment is repeated), the more closely the histogram will approximate the PDF.

The **moments** of the variable v are derived from the PDF. The n -th moment $\langle v^n \rangle$ is defined as

$$\langle v^n \rangle = \int_{-\infty}^{+\infty} v^n P(v)dv \quad (3.2)$$

The first moment $\langle v \rangle$ is the **mean**:

$$\langle v \rangle = \int_{-\infty}^{+\infty} v P(v)dv \quad (3.3)$$

The **variance** is the second moment of the perturbation quantity $v' = v - \langle v \rangle$, and describes the level of variability about this mean.

$$\langle v'^2 \rangle = \int_{-\infty}^{+\infty} (v - \langle v \rangle)^2 P(v)dv \quad (3.4)$$

The **skewness** is the third moment of v' , normalized by the variance:

$$\text{skewness} = \frac{\langle v'^3 \rangle}{\langle v'^2 \rangle^{3/2}} \quad (3.5)$$

A PDF which is symmetric about the mean $\langle v \rangle$ will have zero skewness. All higher odd moments of such a symmetric PDF will also be identically zero. The skewness reveals information about the asymmetry of the PDF. Positive skewness indicates that the PDF has a longer tail for $v - \langle v \rangle > 0$ than for $v - \langle v \rangle < 0$. Hence a positive skewness means that variable v' is more likely to take on large positive values than large negative values. A time series with long stretches of small negative values and a few instances of large positive values, with zero time mean, has positive skewness (Fig. 3.1).

The **kurtosis** (or flatness) is the fourth moment of v' , normalized by the variance:

$$\text{kurtosis} = \frac{\langle v'^4 \rangle}{\langle v'^2 \rangle^2} \quad (3.6)$$

A PDF with longer tails will have a larger kurtosis than a PDF with narrower tails. A time series with most measurements clustered around the mean has low kurtosis, a time series dominated by intermittent extreme events has high kurtosis (Fig. 3.2).

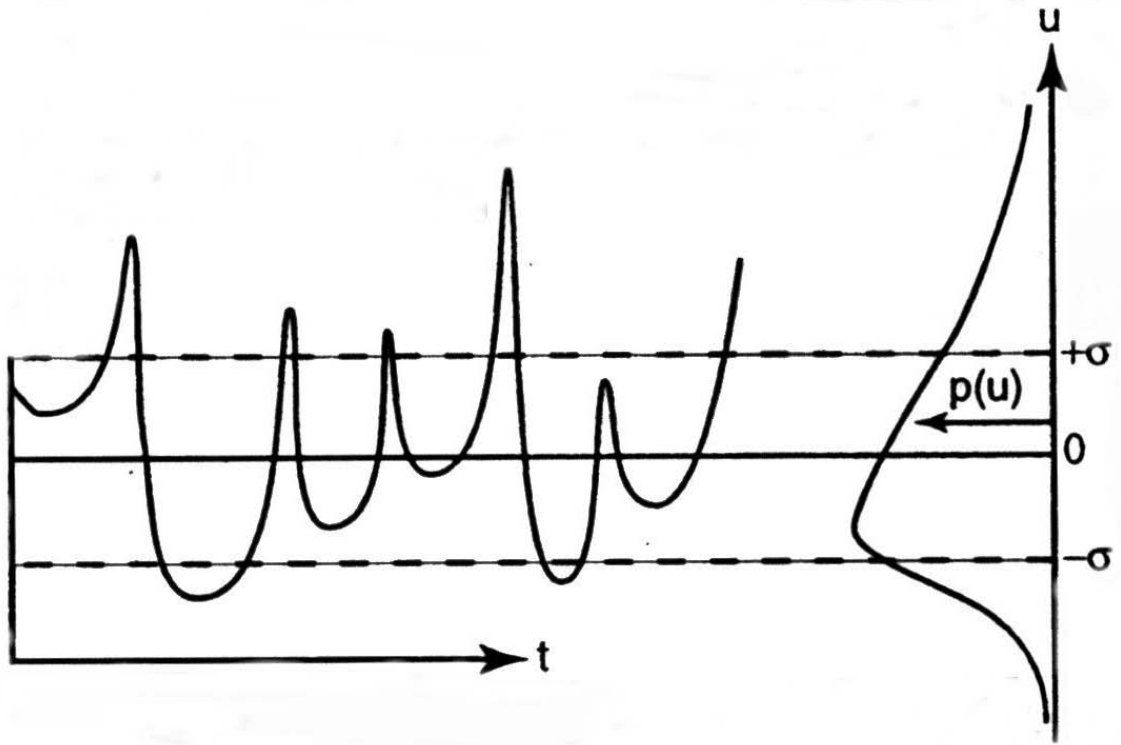


Figure 3.1: Signal with a positive skewness.

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The **characteristic function** $\hat{P}(k)$ is the Fourier transform of the PDF,

$$\hat{P}(k) = \int_{-\infty}^{+\infty} e^{ikv} P(v) dv = \langle e^{ikv} \rangle. \quad (3.7)$$

The characteristic function of the sum of two independent variables is the product of their individual characteristic functions.

If we take the derivative of the above equation with respect to k and evaluate it at the origin ($k = 0$) we see that the n -th moment is related to the derivatives of the characteristic function by,

$$\left. \frac{d^n \hat{P}(k)}{dk^n} \right|_{k=0} = i^n \langle v^n \rangle, \quad (3.8)$$

where $i = \sqrt{-1}$. Then the characteristic function can be written as a Taylor series of the moments,

$$\hat{P}(k) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n \hat{P}(k)}{dk^n} \right|_{k=0} k^n = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle v^n \rangle. \quad (3.9)$$

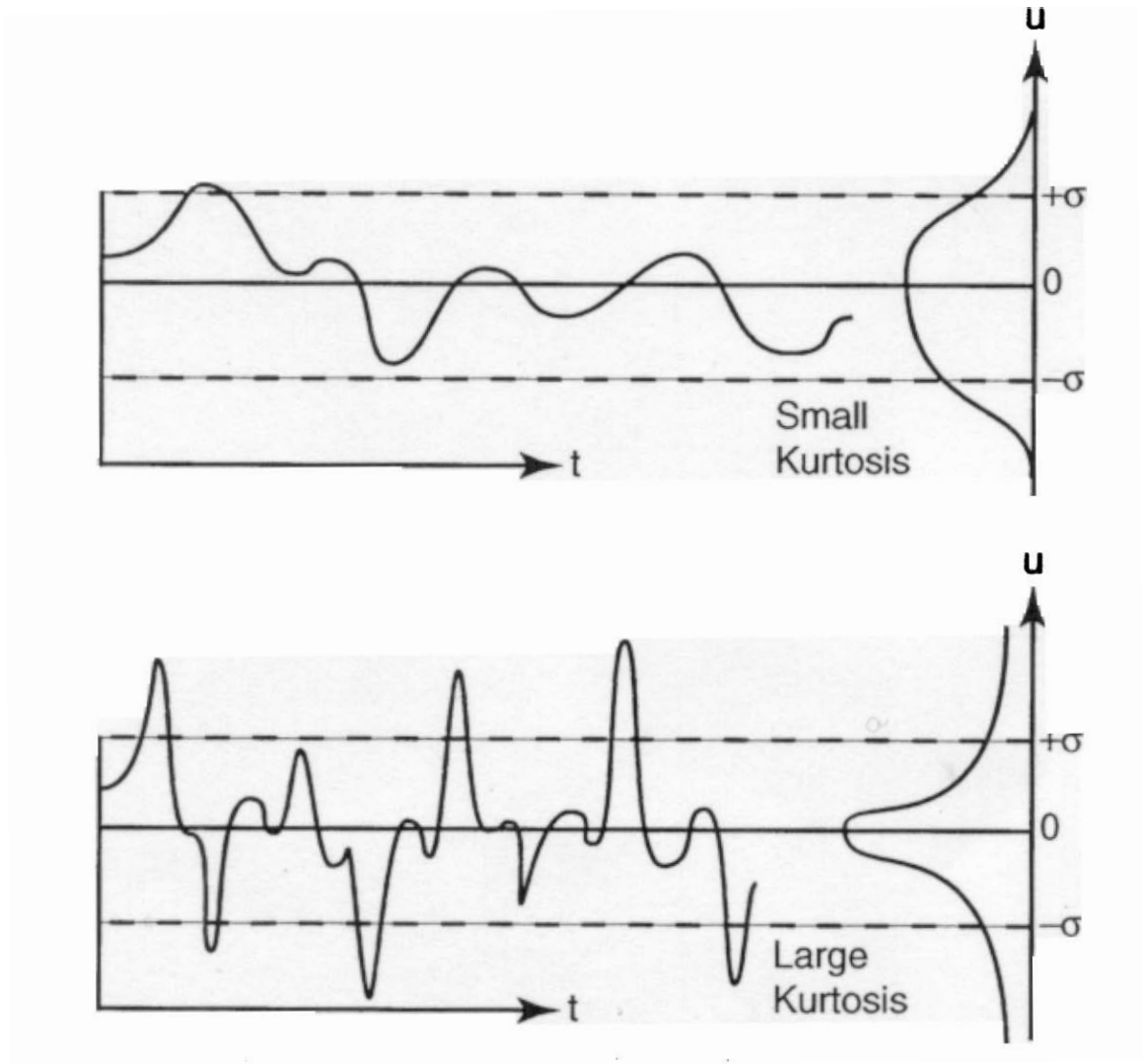


Figure 3.2: Signal with a large kurtosis.

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3.2 Joint probability density functions

Turbulence, of course, involves not one, but several random variables dependent on each other. Therefore, it is necessary to define **joint probability density functions (JPDF)**. For example the JPDF $P_J(u, v)$ of variables u and v is the probability of finding the first random variable between u and $u + du$, and the second one between v and $v + dv$. The integral of P_J over the u, v two-dimensional space is unity by definition,

$$\int_0^\infty \int_{-\infty}^\infty P_J(u, v) du dv = 1, \quad (3.10)$$

and $P_J(u, v)$ is positive definite. We recover the PDF of u by integrating P_J over all values of v , and the PDF of v by integrating over all values of u ,

$$P(u) = \int_{-\infty}^\infty P_J(u, v) dv, \quad P(v) = \int_{-\infty}^\infty P_J(u, v) du. \quad (3.11)$$

The moments of u and v can therefore be obtained from P_J as well. The joint first moment of u and v , $\langle uv \rangle$, is defined as,

$$\langle uv \rangle = \int_{-\infty}^\infty \int_{-\infty}^\infty uv P_J(u, v) du dv. \quad (3.12)$$

The **covariance** of u and v is defined as,

$$\mathcal{C}(u, v) = \langle uv \rangle - \langle u \rangle \langle v \rangle = \langle u'v' \rangle. \quad (3.13)$$

The covariance of u and v normalized by the rms values of u and v (the square roots of their variances) is called the **correlation function** $r(u, v)$ and is used to quantify the degree of correlation between u and v ,

$$r(u, v) = \frac{\langle u'v' \rangle}{\sqrt{\langle u'^2 \rangle} \sqrt{\langle v'^2 \rangle}}. \quad (3.14)$$

For perfectly correlated variables, the correlation function is ± 1 . The covariance is a measure of the asymmetry of the JPDF.

Two variables whose covariance is zero, or equivalently whose correlation is zero, are said to be uncorrelated. Notice that two uncorrelated variables need not to be independent. Statistical independence of two variables requires that the JPDF can be expressed as the product of the individual PDFs,

$$P_J(u, v) = P(u)P(v). \quad (3.15)$$

Thus independent variables are uncorrelated, but the reverse is not true.

These concepts can be extended to three or more variables, but it is rarely the case that one has enough information about the statistics of a flow to measure quantities like $P_J(u, v, w)$.

While PDFs and JPDFs are fundamental to theories of turbulence, one seldom measures or use such quantities. Most often, only the first and the second moments are used to characterize the turbulent flow. Particular importance is given to covariances, which appear in the problem of Reynolds averaging and will be discussed at the end of this lecture.

3.3 Ergodicity and statistical symmetries

Computing ensemble means is a daunting task when dealing with laboratory experiments or numerical simulations, because it requires to repeat experiments or numerical runs over and over. In the case of geophysical flows, the situations is even worse; we cannot ask nature to repeat weather patterns in order to compute our means. Does it mean that a statistical description of turbulence is a nice theoretical idea with no practical relevance? Fortunately not, because we can estimate ensemble means through time and spatial means, under certain assumptions.

- **Stationarity**

Turbulence is **stationary** if all mean quantities (e.g. $\langle v \rangle$, $\langle v^n \rangle$ etc) are invariant under a translation in time. A stationary variable v is **ergodic** if the time average of v converges to the mean $\langle v \rangle$ as the time interval extends to infinity,

$$\frac{1}{T} \int_0^T v(t) dt = \langle v \rangle \text{ as } T \rightarrow \infty. \quad (3.16)$$

(See Frisch, 4.4 for a detailed discussion of ergodicity). In this case a time average is equivalent to an ensemble average.

- **Homogeneity**

Turbulence is **homogeneous** if all the mean quantities are invariant under any spatial translation. Then an ergodic hypothesis allows an ensemble average to be calculated as a spatial average,

$$\frac{1}{L} \int_0^L v(x) dx = \langle v \rangle \text{ as } L \rightarrow \infty. \quad (3.17)$$

- **Isotropy**

Turbulence is **isotropic** if all the mean quantities are invariant under any arbitrary rotation of coordinates.

- **Axisymmetry**

Turbulence is **axisymmetric** if all the mean quantities are invariant under a rotation about one particular axis only (e.g. the z axis for stratified turbulence).

Stationary, isotropic, homogeneous statistics are frequently assumed in theories of turbulence, but are in fact rare in geophysical contexts where stratification and Coriolis break the symmetry with respect to rotation, boundaries break the symmetry with respect to translation, and seasonal and diurnal forcing break the symmetry with respect to time-shifts. However less restrictive assumptions, to be described in the rest of this class, allow progress to be made.

3.4 Central Limit Theorem

Many quantities in turbulence can be thought of as a sum of random variables. For example a net particle displacement in a turbulent flow may be thought as the sum of N random displacements by small eddies; a net velocity may be thought as the sum of N random velocity increments. If we assume the incremental changes x_n all have the same unknown PDF and zero mean value, then the net change in variable is given by,

$$z = \sum_{n=1}^N x_n. \quad (3.18)$$

If all the individual incremental changes are independent and uncorrelated with PDF $P(x)$, the variance of z is given by,

$$\langle z^2 \rangle = \sum_{n=1}^N \sum_{m=1}^N \langle x_n x_m \rangle = N\sigma^2, \quad (3.19)$$

where σ^2 is the variance of all the x_n . (Notice that we are assuming that σ^2 is finite. This is not always the case as for example for power-law PDFs).

The variance of z increases with the size of the ensemble N . As it turns out, it is more convenient to work in terms of the variable $u = N^{-1/2}z$, which has variance σ^2 for all N . We can now determine the PDF of u .

The characteristic function of $P(u)$ is given by

$$\hat{P}_u(k) = \langle e^{iku} \rangle = \langle e^{ik/N^{1/2} \sum_{n=1}^N x_n} \rangle = (\hat{P}(kN^{-1/2}))^N. \quad (3.20)$$

We can expand $\hat{P}(kN^{-1/2})$ as shown in (3.9),

$$\hat{P}(kN^{-1/2}) = 1 - \frac{k^2\sigma^2}{2N} + O(k^3N^{-3/2}). \quad (3.21)$$

For large N we obtain the characteristic function of the sum:

$$\hat{P}_u(k) = \lim_{N \rightarrow \infty} (1 - k^2 \sigma^2 / 2N)^N = e^{-k^2 \sigma^2 / 2}. \quad (3.22)$$

The corresponding PDF is

$$P(u) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-u^2/2\sigma^2}, \quad (3.23)$$

i.e. a Gaussian PDF. Therefore the sum of a large number of identically distributed independent variables has a Gaussian PDF, regardless of the shape of the PDF of the variables themselves. This result is known as the **central limit theorem**.

The sum, normalized by $1/\sqrt{N}$, of N random, independent and identically distributed variables of zero mean and finite variance, σ^2 , is a random variable with a PDF converging to a Gaussian distribution with variance σ^2 .

It is important to remember the conditions under which this statement holds: the individual variables must be independent; N must be large; the variance σ^2 of the PDFs of the individual variables must be finite.

The central limit theorem is most useful as a starting point, because deviations from Gaussian PDFs indicate that some of the assumptions discussed above are broken. This happens often in turbulent flows.

3.5 Intermittency

A signal is said to be intermittent if rare events of large magnitude are separated by long periods with events of low magnitude. Spatial intermittency implies that the signal displays localized regions with events of large magnitude, and wide areas with events of low magnitude. PDFs of intermittent flows are not Gaussian. If turbulence is dominated by coherent structures localized in space and time, then PDFs are not Gaussian. Flows characterized by intermittency are not self-similar.

(Further reading: Tennekes and Lumley, Ch 6; Lesieur, Ch 5; Frisch, Ch 3, Ch 4, and Ch 8 for more on intermittency).

3.6 The Closure Problem

Although it is impossible to predict the detailed motion of each eddy in a turbulent flow, the mean state may not be changing. For example, consider the weather sys-

tem, in which the storms, anti-cyclones, hurricanes, fronts etc. constitute the eddies. Although we cannot predict these very well, we certainly have some skill at predicting their mean state, the climate. For example, we know that next summer will be warmer than next winter, and that in California summer will be drier than winter. We know that next year it will be colder in Canada than in Mexico, although there might be an occasional day when this is not so. We would obviously like to be able to predict the mean climate without necessarily trying to predict or even simulate all the eddies. We might like to know what the climate will be like one hundred years from now, without trying to know what the weather will be like on February 9, 2056, plainly an impossible task. Even though we know what equations determine the system, this task proves to be very difficult because the equations are nonlinear. This is the same problem we discussed at the beginning of the lecture. We seek a statistical description of the turbulent flow, because a detailed description is beyond our grasp. The simplest statistical quantity we might try to predict is the mean flow. However, because of the nonlinearities in the equations, we come up against the closure problem.

The program is to first decompose the velocity field into mean and fluctuating components,

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}' \quad (3.24)$$

Here \mathbf{u} is the mean velocity field, and \mathbf{u}' is the deviation from that mean. The mean may be a time average, in which case $\bar{\mathbf{u}}$ is a function only of space and not time. It might be a time mean over a finite period (e.g a season if we are dealing with the weather). Most generally it is an ensemble mean. Note that the average of the deviation is, by definition, zero; that is $\overline{\mathbf{u}'} = 0$. We then substitute into the momentum equation and try to obtain a closed equation for \mathbf{u} . To visualize the problem most simply, we carry out this program for a model nonlinear system proposed by Geoff Vallis (<http://www.princeton.edu/~gkv/aofd.pdf>) which obeys,

$$\frac{du}{dt} + uu + \nu u = 0. \quad (3.25)$$

The average of this equation is,

$$\frac{d\bar{u}}{dt} + \overline{uu} + \nu\bar{u} = 0. \quad (3.26)$$

The value of the term \overline{uu} is not deducible simply by knowing \bar{u} , since it involves correlations between eddy quantities $\overline{u'u'}$. That is, $\overline{uu} = \bar{u}\bar{u} + \overline{u'u'} \neq \bar{u}\bar{u}$. We can go to next order to try to obtain a value for \overline{uu} . First multiply (3.25) by u to obtain an equation for u^2 , and then average it to yield,

$$\frac{1}{2} \frac{d\overline{uu}}{dt} + \overline{uuu} + \nu\overline{uu} = 0. \quad (3.27)$$

This equation contains the undetermined cubic term \overline{uuu} . An equation determining this would contain a quartic term, and so on in an unclosed hierarchy. Most current

methods of closing the hierarchy make assumptions about the relationship of (n+1)-th order terms to n-th order terms, for example by supposing that,

$$\overline{uuuu} = \overline{uu} \overline{uu} - \alpha \overline{uuuu}, \quad (3.28)$$

where α is some parameter. Such assumptions require additional, and sometimes dubious, reasoning. Nobody has been able to close the system without introducing physical assumptions not directly deducible from the equations of motion.

3.7 The Reynolds equations

Let us repeat the averaging procedure for the full Boussinesq equations. We start with the momentum equations,

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} + f \hat{\mathbf{z}} \times \bar{\mathbf{u}} = \bar{b} \hat{\mathbf{z}} - \frac{1}{\rho_0} \nabla \bar{p} + \nu \nabla^2 \bar{\mathbf{u}} - \overline{(\mathbf{u}' \cdot \nabla) \mathbf{u}'}, \quad (3.29)$$

The extra term on the right hand side represent the effect of eddy motions on the mean flow. If the average operator is a time average over some time T , then eddy motions are those motions with time scales shorter than T . If the average operator is a spatial average over some scale L , then eddy motions are those motions with spatial scales shorter than L . If the average operator is an ensemble mean, then the eddy motions are those motions that change in every realizations, regardless of their scale, i.e. they represent the unpredictable or turbulent part of the flow.

Using the continuity equation,

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \nabla \cdot \bar{\mathbf{u}} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{u}' = 0, \quad (3.30)$$

we can rewrite the averaged momentum equation as,

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} + f \hat{\mathbf{z}} \times \bar{\mathbf{u}} = \bar{b} \hat{\mathbf{z}} - \frac{1}{\rho_0} \nabla \cdot [\bar{p} \mathbf{I} - \rho_0 \nu \nabla \bar{\mathbf{u}} + \rho_0 \overline{\mathbf{u}' \mathbf{u}'}]. \quad (3.31)$$

\mathbf{I} is the unit matrix. These are the so-called **Reynolds momentum equation** and the eddy flux $\rho_0 \overline{\mathbf{u}' \mathbf{u}'}$ represent the **Reynolds stress tensor** due to fluctuations in velocity field.

We can similarly decompose the buoyancy equation into a mean and a fluctuating component, $b = \bar{b} + b'$, and write an equation for the mean component by substituting back into the buoyancy equation,

$$\frac{\partial \bar{b}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \bar{b} = -\nabla \cdot [-\kappa \nabla \bar{b} + \overline{\mathbf{u}' b'}]. \quad (3.32)$$

The eddy term $\overline{\mathbf{u}'b'}$ represent the **Reynolds eddy flux** of buoyancy.

The problem of turbulence is often that of finding a representation of such Reynolds stress and flux terms in terms of mean flow quantities. However, it is not at all clear that a general solution (or parameterization) exists, short of computing the terms explicitly.

3.8 Eddy viscosity and eddy diffusivity

The simplest closure for the Reynolds stress terms is one which relates $\overline{\mathbf{u}'\mathbf{u}'}$ to the mean flow, by assuming a relation of the form,

$$\overline{\mathbf{u}'\mathbf{u}'} = -\nu_T \nabla \bar{\mathbf{u}}, \quad (3.33)$$

where ν_T is the **eddy viscosity**. With such a closure the Reynolds stress term takes the same form as the mean viscosity term, but with a different viscosity. In essence, this closures states that turbulent eddies are similar to molecular motions that constantly act to redistribute and homogenize momentum. Similarly, for the tracer flux term we can define an **eddy diffusivity**

$$\overline{\mathbf{u}'b'} = -\kappa_T \nabla \bar{b}. \quad (3.34)$$

This eddy viscosity/diffusivity closure is the most commonly used in modeling and interpretation of geophysical observations. At the crudest level κ_T and ν_T are assumed to be constants; in more sophisticated models they are functions of the large scale flow. However, an eddy viscosity/diffusivity closure is rarely appropriate.

(Further reading: Chapter 1 of McComb)