Quasi-equilibrium Theory of Small Perturbations to Radiative-Convective Equilibrium States

- See *Quasi-Equilibrium Dynamics of the Tropical Atmosphere* paper on course web site
- Free troposphere assumed to have moist adiabatic lapse rate (s^* does not vary with height
- Boundary layer quasi-equilibrium applies

Basis of statistical equilibrium physics

- Dates to Arakawa and Schubert (1974)
- Analogy to continuum hypothesis: Perturbations must have space scales >> intercloud spacing
- TKE consumption by convection ~ CAPE generation by large scale
- Numerical models on the verge of simulating clouds + large-scale waves
- We further assume convective criticality

Implications of the moist adiabatic lapse rate for the structure of tropical disturbances

- Approximate moist adiabatic condition as that of constant saturation entropy:

$$s^* = c_p \ln\left(T / T_0 \right) - R_d \ln\left(p / p_0 \right) + \frac{L_v q^*}{T}$$

- Assume hydrostatic perturbations:

$$\frac{\partial \phi'}{\partial p} = -\alpha'$$

- Maxwell's relation:

$$\alpha' = \left(\frac{\partial \alpha}{\partial s^*} \right)_p s^{*\prime} = \left(\frac{\partial T}{\partial p} \right)_{s^*} s^{*\prime}$$

- Integrate:

$$\phi' = \phi_b'(x, y, t) + \left(\bar{T}(x, y, t) - T \right) s^{*\prime}$$

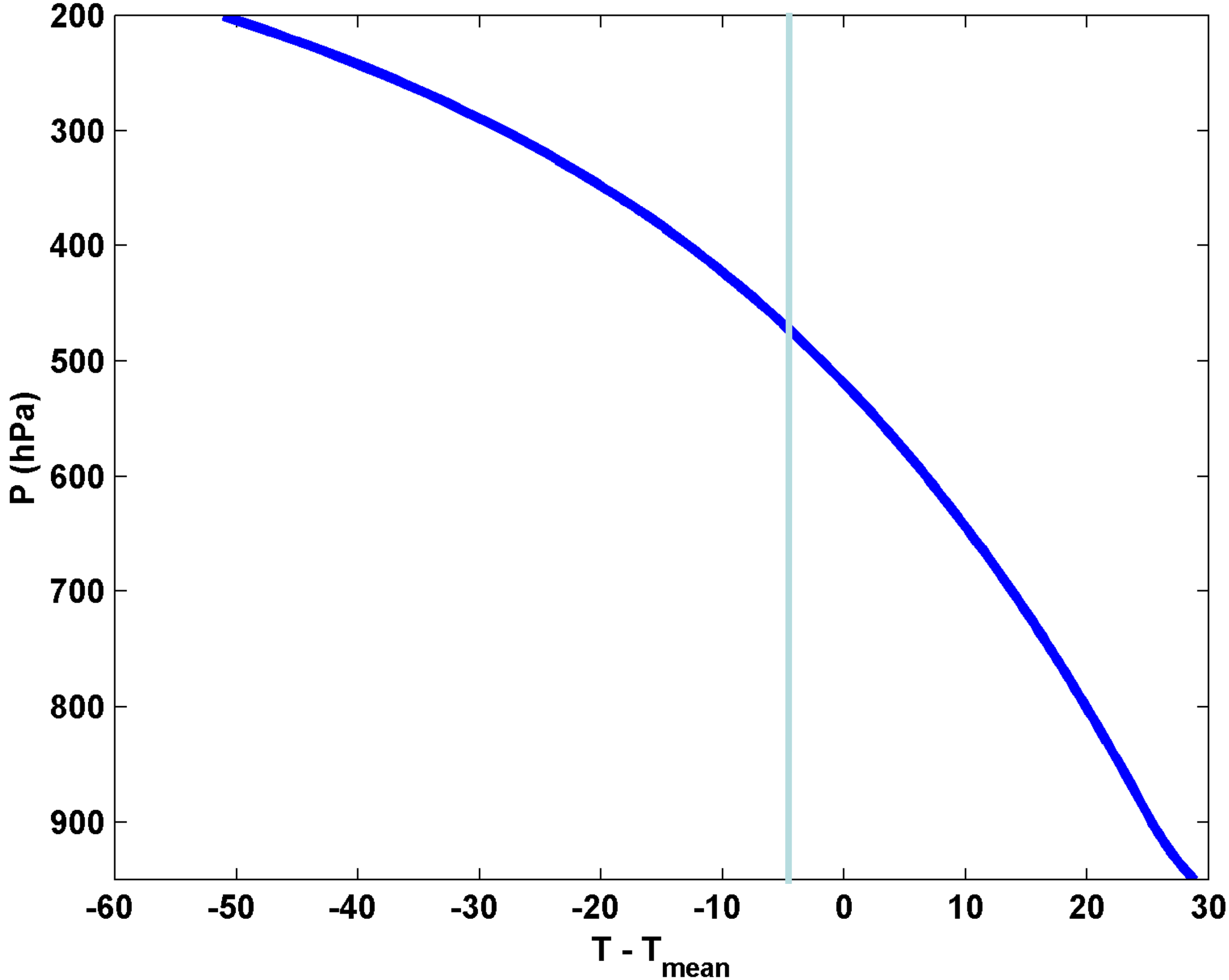
Only barotropic and first baroclinic mode survive

This implies, through the linearized momentum equations, e.g.

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + fv$$

that the horizontal velocities may be partitioned similarly:

$$u = u_b(x,y,t) + \left(\overline{T}(x,y,t) - T\right)u^*(x,y,t);$$
$$v = v_b(x,y,t) + \left(\overline{T}(x,y,t) - T\right)v^*(x,y,t).$$


P (hPa)
200
300
400
500
600
700
800
900
-60
-50
-40
-30
-20
-10
0
10
20
30
T - T mean

Implications for vertical structure of vertical velocity

$$\frac{\partial \omega}{\partial p} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

Integrate:

$$\omega = \left(p_0 - p \right) \left(\frac{\partial u_b}{\partial x} + \frac{\partial v_b}{\partial y} \right) - \left(\left(p_0 - p \right) \overline{T} - \int_p^{p_0} T dp' \right) \left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \right).$$

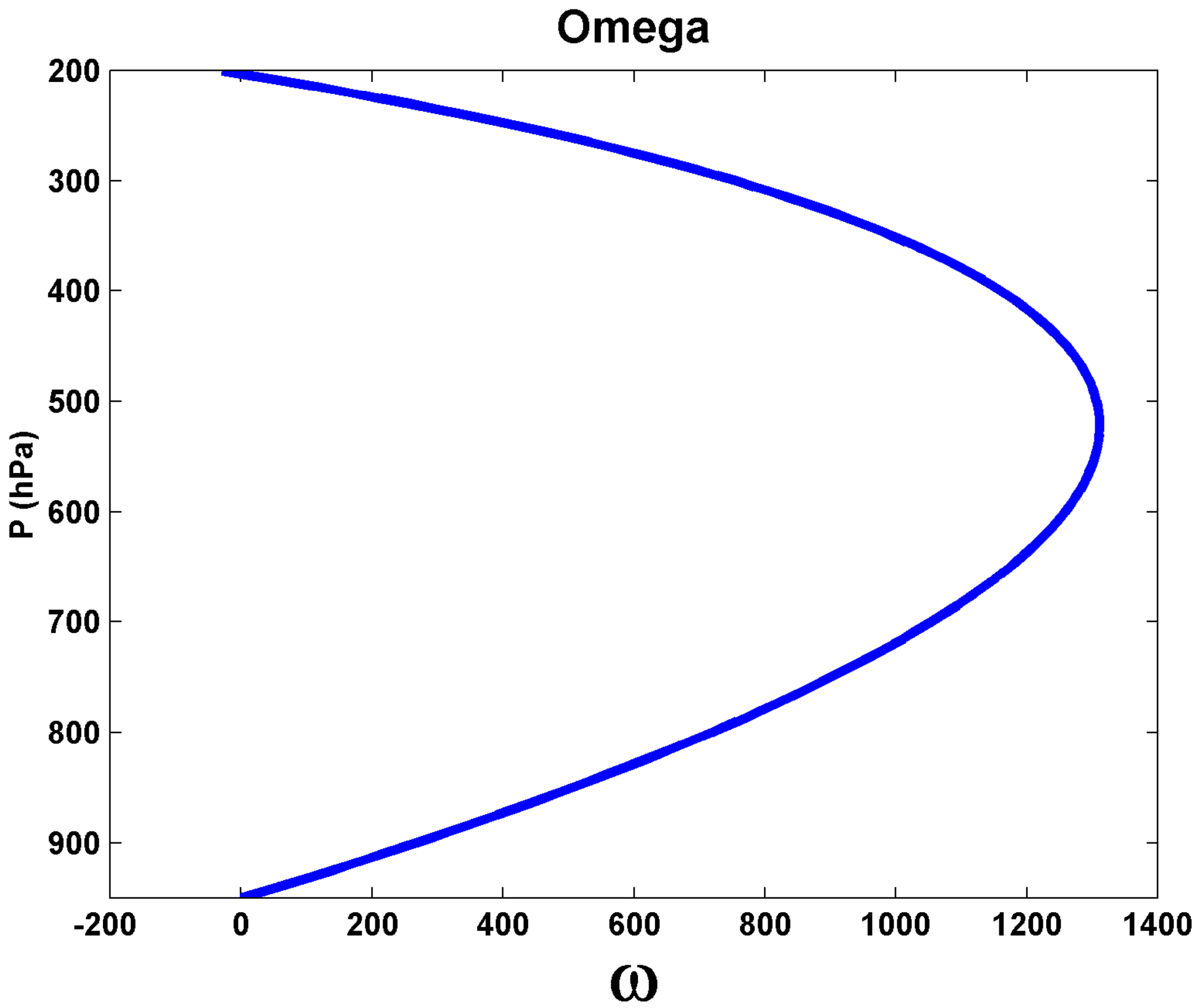
At tropopause:

$$\omega_t = \left(p_0 - p_t \right) \left(\frac{\partial u_b}{\partial x} + \frac{\partial v_b}{\partial y} \right)$$

This implies that if a rigid lid is imposed at the tropopause, the divergence of the barotropic velocities must vanish and the barotropic components therefore satisfy the barotropic vorticity equation:

$$\frac{\partial \eta_b}{\partial t} = -\mathbf{V} \bullet \nabla \eta_b ,$$

$$\eta_b \equiv \hat{k} \bullet \nabla \times \mathbf{V}_b + 2\Omega \sin\theta$$



Omega
200
300
400
500
600
700
800
900
P (hPa)
-200
0
200
400
600
800
1000
1200
1400
ω

Feedback of Air Motion on (virtual) Temperature

- Convection cannot change vertically integrated enthalpy, $k = c_p T + L_v q$
- Then neglecting surface fluxes, radiation, and horizontal advection,

$$\frac{\partial}{\partial t}\int k\,dp = -\int \omega \frac{\partial h}{\partial p} dp,$$

- Neelin and Held (1987): This function is negative for upward motion

- Upward motion is associated with column moistening:

$$\int c_p \frac{\partial T}{\partial t} dp = \frac{\partial}{\partial t} \int k \, dp - \int L_v \frac{\partial q}{\partial t} dp$$

→ **Ascent leads to cooling**

- Yano and Emanuel, 1991:

$$N_{eff}^2 = \left(1 - \varepsilon_p\right) N^2$$

Prediction: Inviscid, small amplitude perturbations under rigid lid: Shallow water solutions with reduced equivalent depth

Quasi-Linear β Plane System , Neglecting Barotropic Mode

$$\frac{\partial u}{\partial t} = \left(T_s - \overline{T}\right)\frac{\partial s*}{\partial x} + \beta yv - ru$$

$$\frac{\partial v}{\partial t} = \left(T_s - \overline{T}\right)\frac{\partial s*}{\partial y} - \beta yu - rv$$

$$\frac{\partial s*}{\partial t} = \frac{\Gamma_d}{\Gamma_m}\left(\dot{Q}_{rad} + \frac{\partial s_d}{\partial z}\left(\varepsilon_p M - w\right)\right)$$

$$h\frac{\partial s_b}{\partial t} = C_k \, |\mathbf{V}| \left(s_0 * - s_b\right) - \left(M - w\right)\left(s_b - s_m\right)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{w}{H}=0$$

Quasi-Equilibrium Assumption:

$$\frac{\partial s_b}{\partial t}=\frac{\partial s^*}{\partial t}$$

Gives closure for convective mass flux, M

System closed except for specification of

$$\dot{Q}_{rad},\ s_0{}^*,\ s_m,\ \varepsilon_p$$

Additional Approximations:

- **Boundary Layer QE** (Raymond, 1995): Neglect $h \partial s_b / \partial t$, gives simpler expression for M

$$M = w + C_k \,|\mathbf{V}|\, \frac{s_0 * - s_b}{s_b - s_m}$$

- **Weak Temperature Approximation** (Sobel and Bretherton, 2000): Neglect $\partial s^* / \partial t$ (Over-determined system, ignore momentum equation for irrotational flow)

Important Feedbacks:

- **Wind-Induced Surface Heat Exchange** (WISHE) Coupling of surface enthalpy flux to wind perturbations (Neelin et al. 1987, Emanuel, 1987)
- **Moisture-Convection Feedback**: Dependence of s_m on M and/or ε_p on $s^* - s_m$
- **Cloud-Radiation Feedback**: Dependence of $\dot{Q}_{rad}$ on M or $s^* - s_m$

- **Ocean-Atmosphere Feedback** (e.g. ENSO): Feedback between perturbation surface wind and ocean surface temperature, as represented by s_0 *

Simple Example:

- Ignore perturbations of $\dot{Q}_{rad}$
- Ignore fluctuations of ε_p
- Make boundary layer QE approximation
- Fully linearize surface fluxes:

$$\overline{|\mathbf{V}|} = \sqrt{\bar{U}^2 + u*^2}$$

$$|\mathbf{V}|' = \frac{\bar{U}u'}{\overline{|\mathbf{V}|}}$$

Introduce scalings:

First define a merdional scale, L_y :

$$L_y^{\,4} = \frac{\Gamma_d}{\Gamma_m}\left(T_s - \overline{T}\right) H \frac{\partial s_d}{\partial z} \frac{1-\varepsilon_p}{\beta^2}$$

Then let

$$x \rightarrow a\, x \qquad\qquad y \rightarrow L_y\; y$$

$$t \rightarrow \frac{a}{\beta L_y^{\,2}}\, t \qquad\qquad u \rightarrow \frac{aC_k \overline{|\mathbf{V}|}}{H}\, u$$

$$v \rightarrow \frac{L_y C_k \overline{|\mathbf{V}|}}{H}\, v \qquad\qquad s^* \rightarrow \frac{aC_k \overline{|\mathbf{V}|\,\beta L_y^{\,2}}}{H\left(T_s - \overline{T}\right)}\, s^*$$

Separate scalings for ocean temperature and lower tropospheric entropy:

$$s_o{}^* \rightarrow \frac{1-\varepsilon_p}{\varepsilon_p}\left(\overline{s_b - s_m}\right) s_0$$

$$s_m \rightarrow \frac{1-\varepsilon_p}{\varepsilon_p}\frac{\left(\overline{s_b - s_m}\right)^2}{\left(\overline{s_0{}^* - s^*}\right)} s_m$$

Nondimensional parameters:

$$\alpha \equiv \frac{1-\varepsilon_p}{\varepsilon_p} \frac{aC_k}{H} \frac{\bar{U}}{\overline{|\mathbf{V}|}} \frac{\left(\overline{s_0 * - s *}\right)}{\left(\overline{s * - s_m}\right)} \qquad (WISHE)$$

$$\mathcal{R} \equiv \frac{ra}{\beta L_y^{\,2}} \qquad (Rayleigh\ \ friction)$$

$$\chi \equiv \frac{1-\varepsilon_p}{\varepsilon_p} \frac{aC_k \,|\mathbf{V}|\, \beta L_y^{\,2}}{H\left(T_s - \bar{T}\right)} \frac{\left(\overline{s_0 * - s *}\right)}{\left(\overline{s * - s_m}\right)^2} \qquad (surface\ damping)$$

$$\delta = \left(\frac{a}{L_y}\right)^2 \qquad (zonal\ geostropy)$$

Nondimensional Equations:

$$\frac{\partial u}{\partial t} = \frac{\partial s}{\partial x} + yv - \mathcal{R}u$$

$$\frac{\partial v}{\partial t} = \delta\left(\frac{\partial s}{\partial y} - yu - \mathcal{R}v\right)$$

$$\frac{\partial s}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \alpha u + s_0 + s_m - \chi s$$

Steady System with $\mathcal{R} = s_m = 0$:

$$\frac{\partial s}{\partial x} + \alpha y \frac{\partial s}{\partial y} - \chi y^2 s = -y^2 s_0$$

Similar to Gill Model, but forcing is directly in terms of SST (s_0), not latent heating

For SST of the form

$$s_0 = \mathbf{RE}\left[G(y) e^{ikx} \right]$$

there are solutions of the form

$$s = \mathbf{RE}\left\lfloor J(y) e^{ikx} \right\rfloor,$$

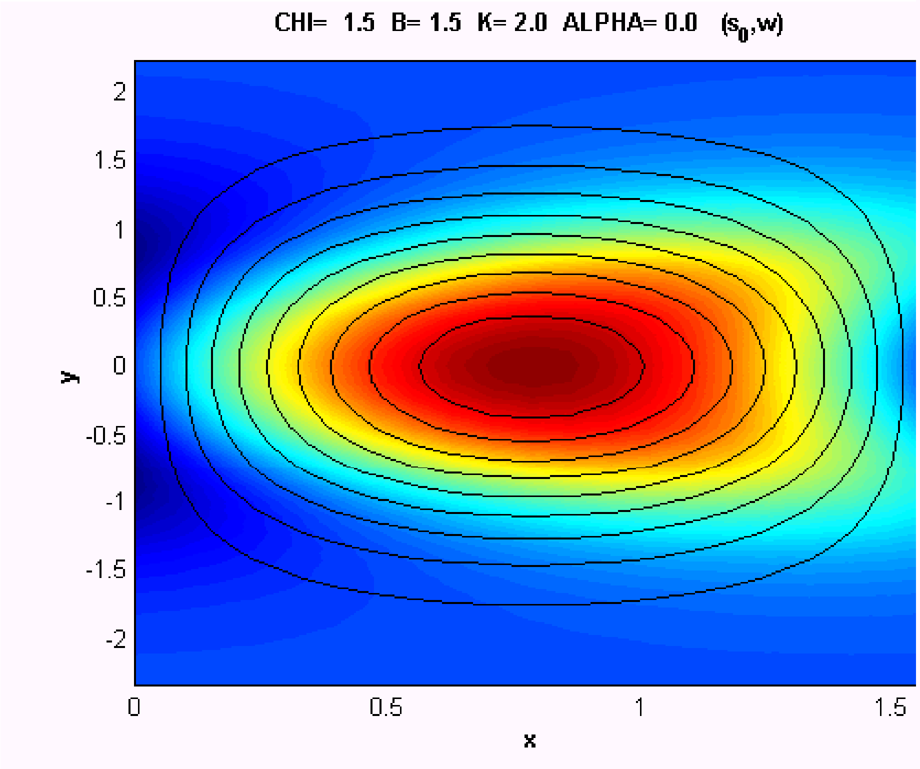
where

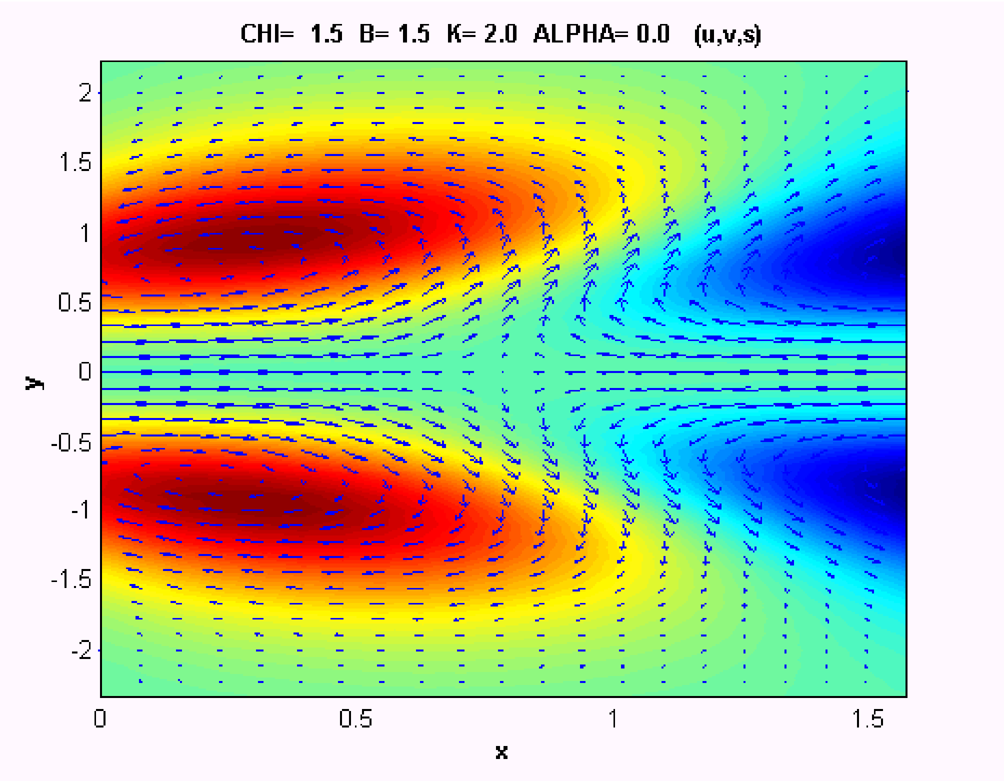
$$J(y) = y^{-ik/\alpha} e^{\chi y^2/2\alpha} \int_0^y G u^{1+ik/\alpha} e^{-\chi u^2/2\alpha} du$$

Example:

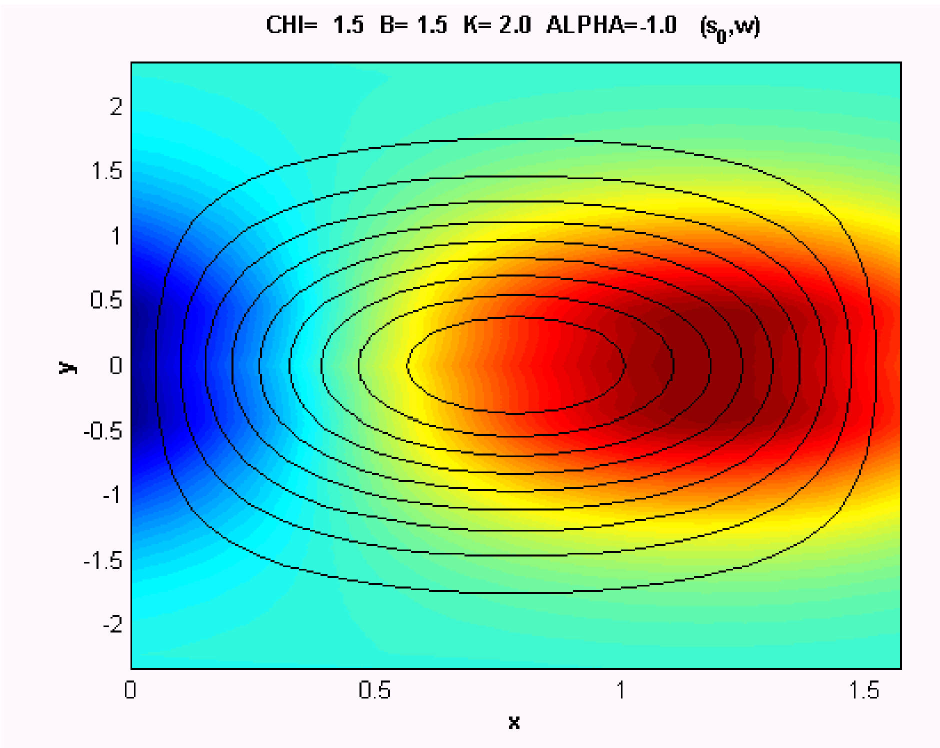
$$G = e^{-by^2}$$

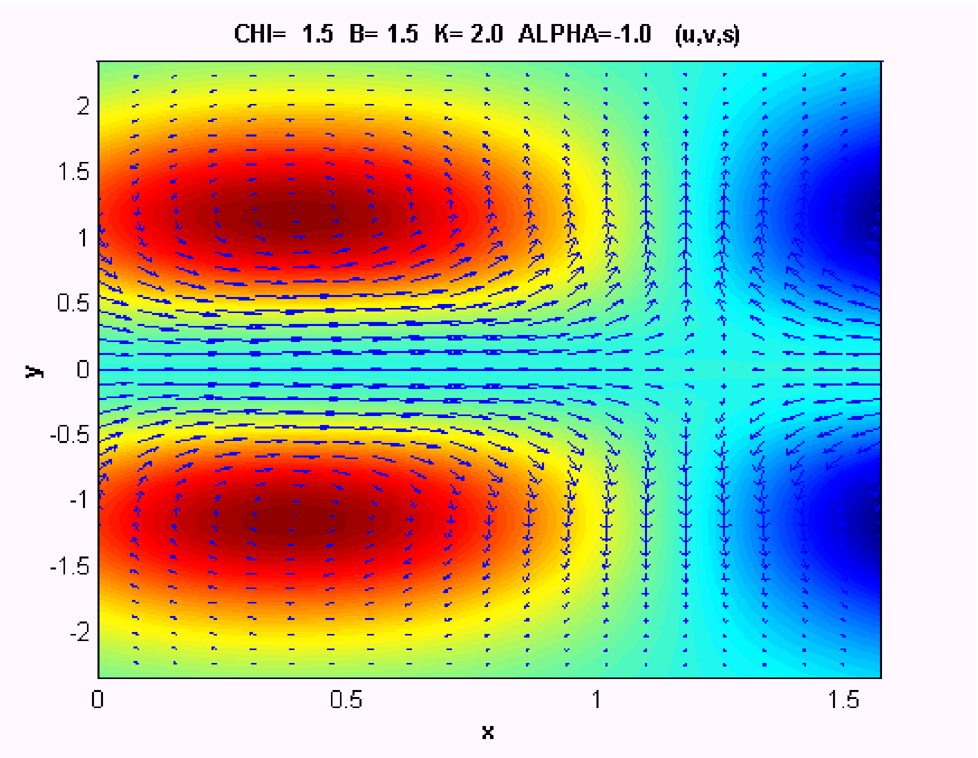
$$\alpha = 0, \quad k = 2, \quad b = 1.5, \quad \chi = 1.5$$





$$\alpha = -1, \quad k = 2, \quad b = 1.5, \quad \chi = 1.5$$





Basic linear wave dynamics on the equatorial β plane

Omit damping and WISHE terms from linear nondimensional equations:

$$\frac{\partial u}{\partial t} = \frac{\partial s}{\partial x} + yv$$

$$\frac{\partial v}{\partial t} = \delta\left(\frac{\partial s}{\partial y} - yu\right)$$

$$\frac{\partial s}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

Fully equivalent to the shallow water equations on a β plane

Eliminate *s* and *u* in favor of *v*:

$$\frac{\partial}{\partial t}\left\{\frac{\partial^2 v}{\partial t^2}-\frac{\partial^2 v}{\partial x^2}-\delta\frac{\partial^2 v}{\partial y^2}+\delta y^2 v\right\}-\delta\frac{\partial v}{\partial x}=0$$

Let $v=V(y)e^{ikx-i\omega t}$

$$\rightarrow\frac{d^2V}{dy^2}+\left(\frac{\omega^2-k^2}{\delta}-\frac{k}{\omega}-y^2\right)V=0$$

Boundary conditions: *V* well behaved at $y \to \pm\infty$

Solution in terms of discrete parabolic cylinder functions D_n :

$$v = D_n(y),$$

$$where\ D_n = e^{-2y^2}\left[1, 2y, 4y^2 - 2, \ldots\right]$$

provided ω satisfies the dispersion relation

$$\boxed{\frac{\omega^2 - k^2}{\delta} - \frac{k}{\omega} = 2n + 1}$$

There is, in addition, another mode satisfying v=0 everywhere. From first and third linear equations:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.$$

$$\textit{Satisfied by} \quad u = F\left(x - t\right).$$

Eastward-propagating, nondispersive equatorially trapped Kelvin wave

Note that this happens to satisfy derived dispersion relation when n= -1.

There are three roots of the general dispersion relation:

$$n=0: \quad \frac{\omega^2-k^2}{\delta}-\frac{k}{\omega}-1=0$$

$$Factor: \quad \left(\frac{\omega-k}{\delta}-\frac{1}{\omega}\right)(\omega+k)=0$$

$$\omega=-k \text{ root not allowed } (\text{does not satisfy BCs})$$

$$\omega=\frac{1}{2}\left(k\pm\sqrt{k^2+4\delta}\right)$$

Mixed Rossby-Gravity Waves (MRG)

For n ≥ 1, two well defined limits:

$$1.\ |\omega| << |k|:$$

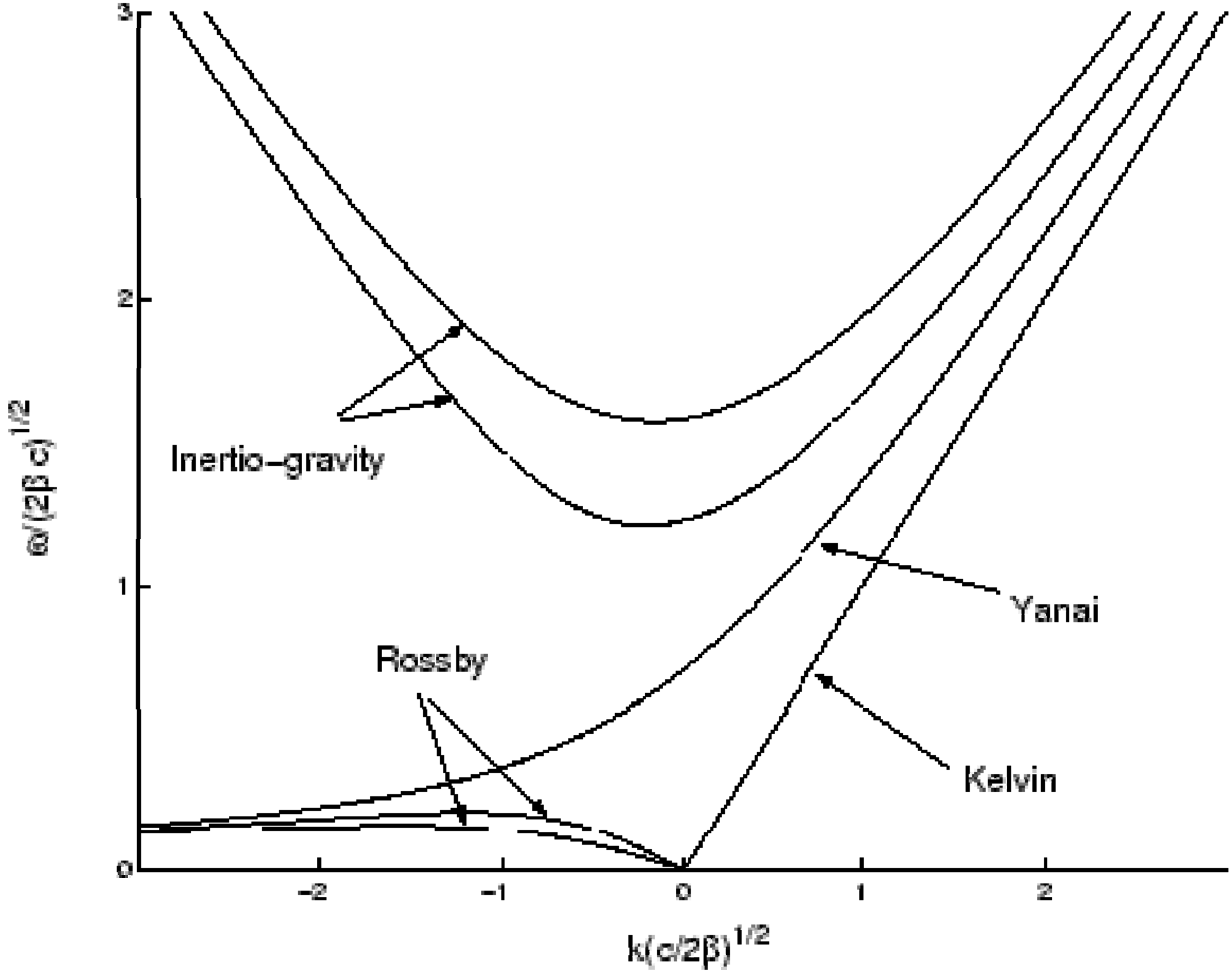
$$\omega \cong -\frac{k}{2n+1+k^2/\delta}$$

Planetary Rossby waves

$$2.\quad |\omega| >> |k|:$$

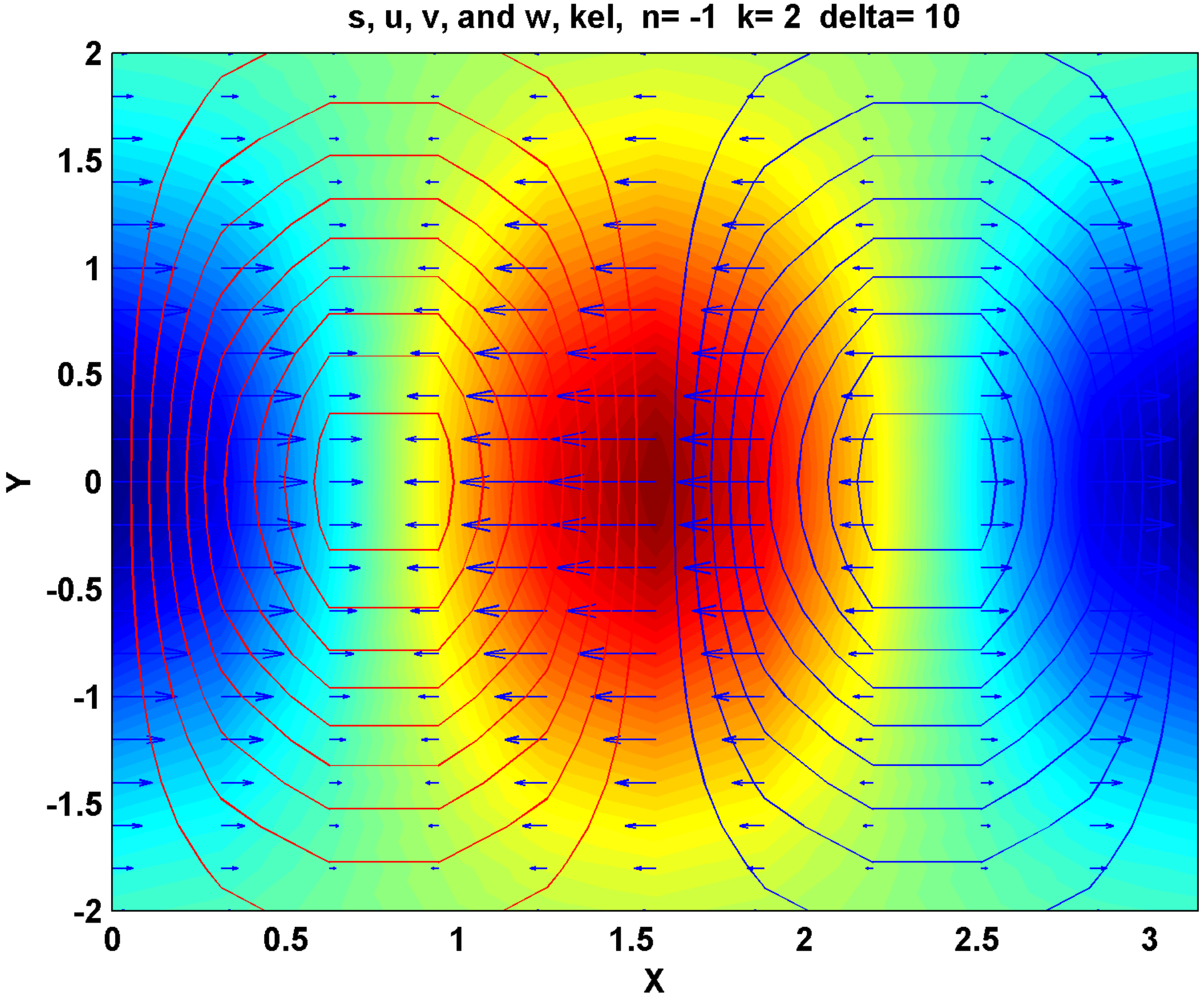
$$\omega^2 \cong k^2 + \delta(2n+1)$$

Inertia-gravity waves

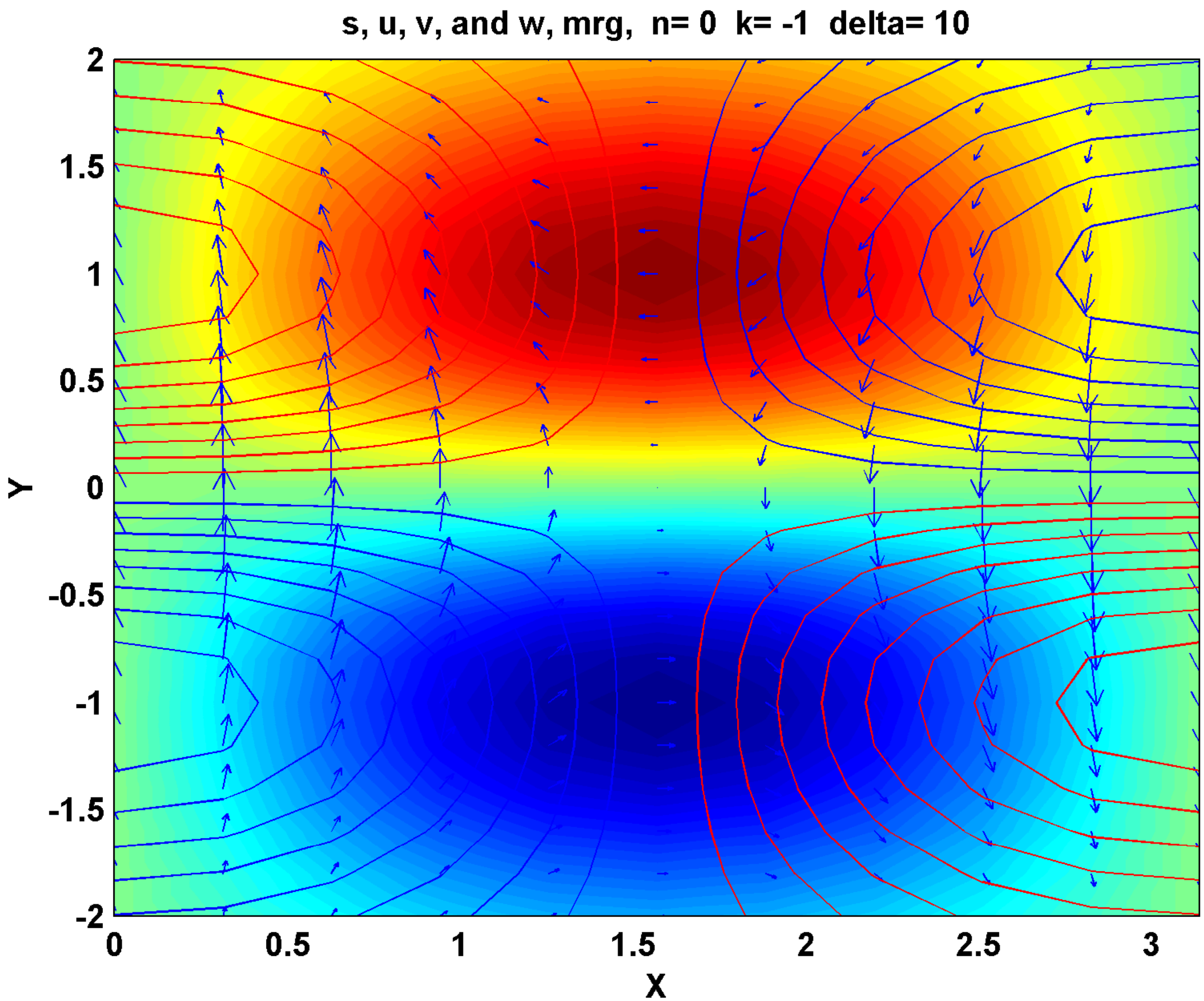


3
2
1
0
ω/(2β c)^1/2
Inertio–gravity
Yanai
Rossby
Kelvin
−2
−1
0
1
2
k(c/2β)^1/2

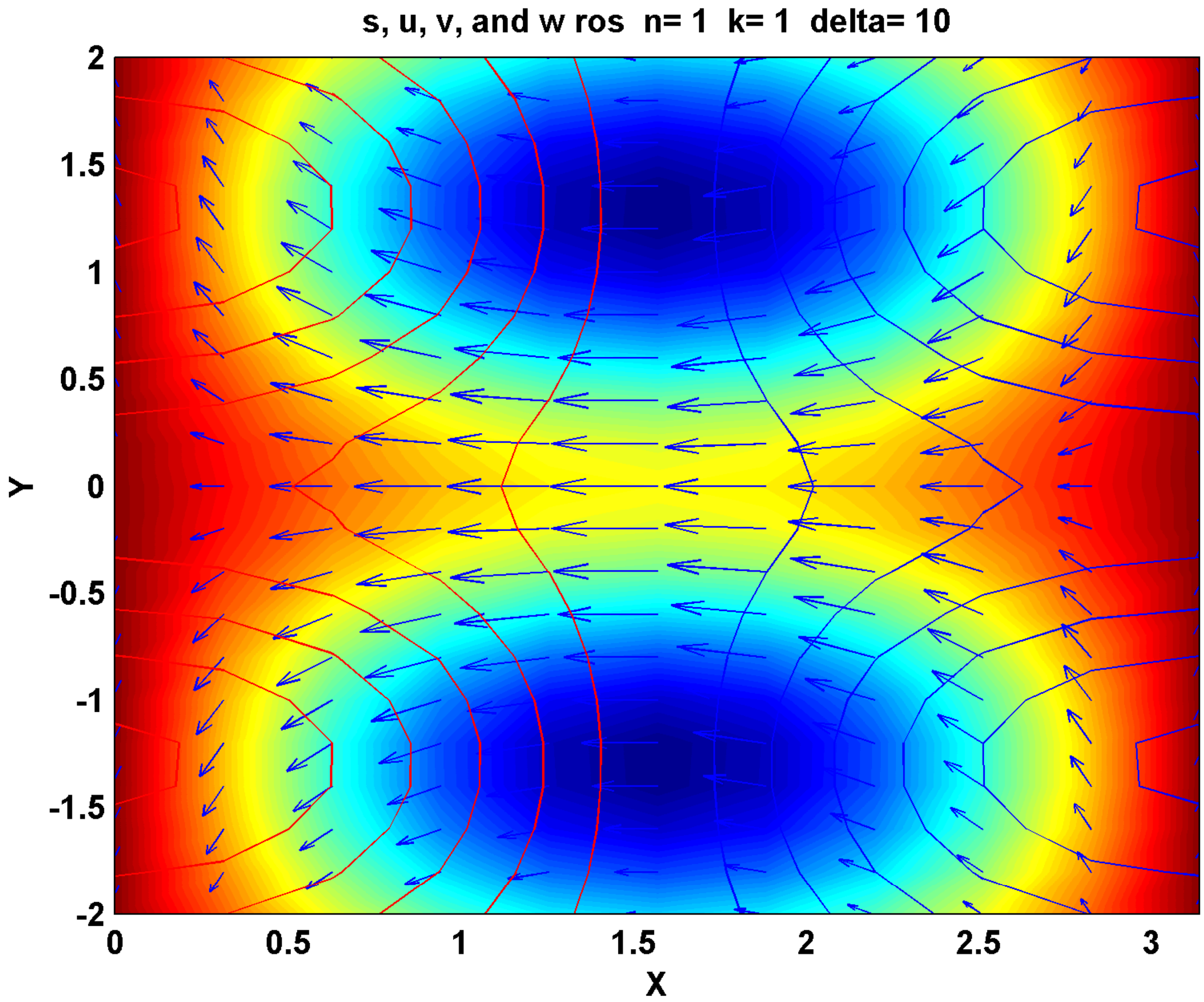
Kelvin wave



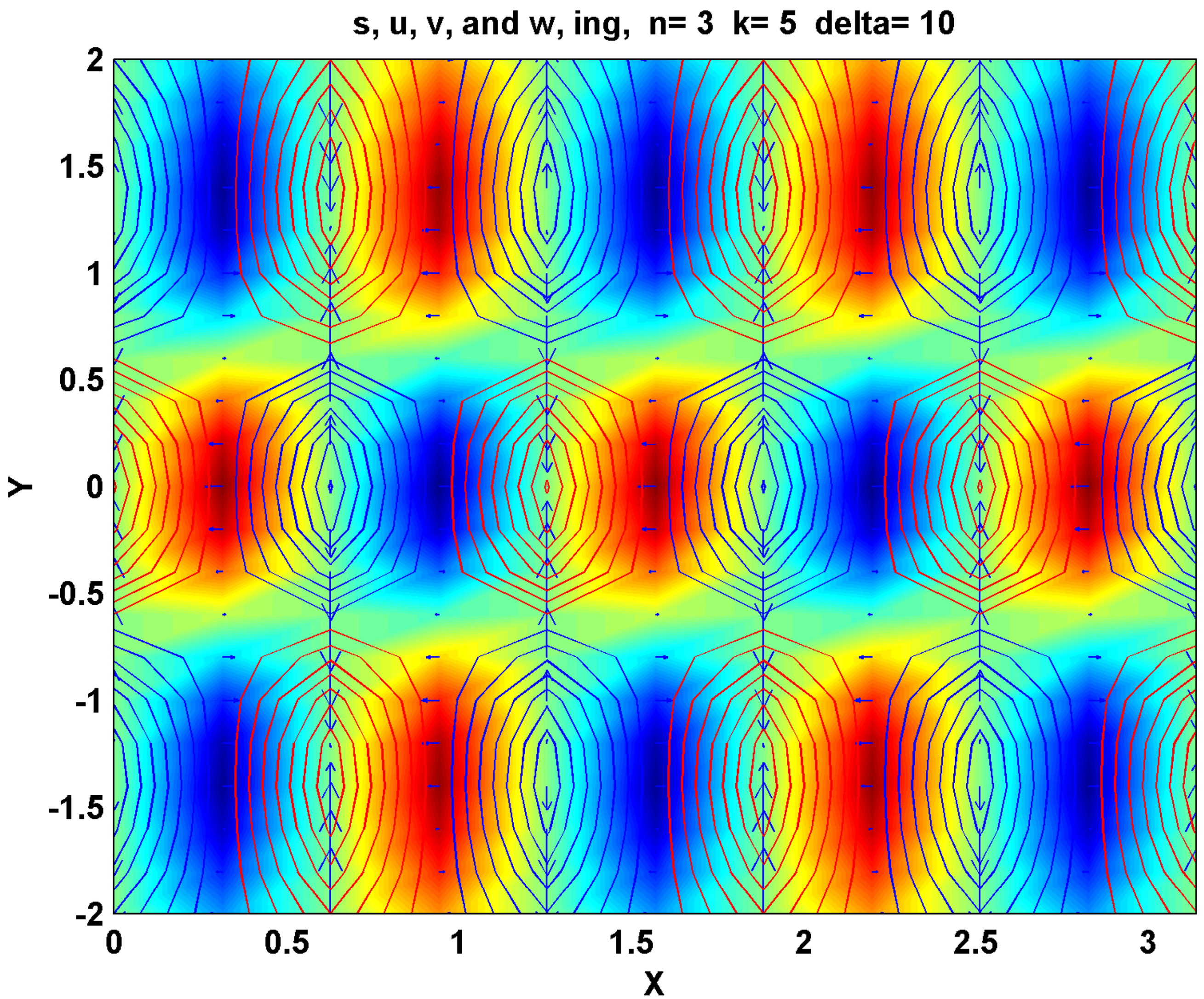
Mixed Rossby-Gravity



Rossby



Inertia-gravity



12.811 Tropical Meteorology
Spring 2011