Surface gravity waves

For a homogeneous layer of fluid with a free surface at $z = \eta(x, y, t)$ and flat bottom at $z = -H$, we rewrite the pressure as $p = -\rho_0 g z + \rho_0 P(x, y, z, t)$ and the equations of motion become

$$\frac{\partial}{\partial t}\mathbf{u} + \boldsymbol{\zeta} \times \mathbf{u} = -\nabla(P + \frac{1}{2}\mathbf{u} \cdot \mathbf{u})$$
$$\nabla \cdot \mathbf{u} = 0$$

Irrotational flow:

When the vorticity $\boldsymbol{\zeta} = \nabla \times \mathbf{u}$ is initially zero, it remains zero. We can see this from the vorticity equation derived by taking the curl of the momentum equation

$$\frac{\partial}{\partial t}\boldsymbol{\zeta} + \nabla \times (\boldsymbol{\zeta} \times \mathbf{u}) = 0$$

Therefore, if $\boldsymbol{\zeta}(\mathbf{x}, 0) = 0$, the vorticity will remain zero thereafter. In that case, the velocity is given by the gradient of a potential function

$$\mathbf{u} = -\nabla\phi$$

We can define a scalar function

$$\phi(\mathbf{x}_0) - \phi(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{u} \cdot d\ell$$

with the integral taken along a path joining the two points. The integral is path-independent, since the integral along a closed path $\oint \mathbf{u} \cdot d\ell = 0$ by Stokes' theorem. Thus ϕ with this definition is indeed a scalar function, and its derivatives with respect to $\mathbf{x}$ give the velocity components.

The equations in the interior of the fluid simplify to

$$-\nabla\frac{\partial}{\partial t}\phi = -\nabla(P + \frac{1}{2}|\nabla\phi|^2)$$
$$or$$
$$\frac{\partial}{\partial t}\phi = P + \frac{1}{2}|\nabla\phi|^2$$
$$\nabla^2\phi = 0$$

The first equation tells us how the pressure varies given the potential; it's the second equation that determines the structure of the field. The wave part of the dynamics doesn't appear obvious in Laplace's equation; instead it shows up in the boundary conditions.

Boundary conditions:

The lower boundary condition is straightforward: it states that the normal component of velocity vanishes

$$w = -\frac{\partial\phi}{\partial z} = 0 \quad at \quad z = -H$$

Note that this has a broader implication. It implies that a particle of fluid on the bottom can move along the bottom but not off of it. In molecular terms, the condition simply states that the mean upward and downward velcotoes are equal. The fact that molecules can migrate away from the surface (and tracer material as well) is connected to the diffusive part of the dynamics, not the molecular mean velocity $\mathbf{u}$.

At the top surface, $z = \eta(x, y, t)$, we apply the same argument that particles remain at the interface; however, we must now account for the motion of the interface as well. We find

$$\frac{\partial}{\partial t}\eta + \mathbf{u}(x, y, \eta, t) \cdot \nabla\eta = w(x, y, \eta, t)$$

or

$$\frac{D}{Dt}(\eta - z) = 0 \quad with \quad \frac{D}{Dt} = \frac{\partial}{\partial t} - \nabla\phi \cdot \nabla$$

Finally, we apply a dynamic condition at the surface that the pressure of the fluid must equal the pressure of the air above

$$p_a = -\rho_0 g\eta + \rho_0 P(x, y, \eta, t) \quad \Rightarrow \quad P(x, y, \eta, t) = g\eta(x, y, t) + p_a/\rho_0$$

Using the Bernouilli equation allows us to write this condition in terms of the potential and the surface elevation

$$\frac{\partial}{\partial t}\phi - \frac{1}{2}|\nabla\phi|^2 = g\eta + p_a/\rho_0 \quad at \quad z = \eta$$

We could combine the kinematic and dynamic conditions to express the upper boundary condition solely in terms of ϕ; however, we will leave the two fields explicitly.

LINEARIZED EQNS:

We can linearize the equations by assuming that velocities are small compared to the phase speed and chenges in elevation are small compared tot he wavelength or the depth of the fluid. Linearization alters the upper boundary conditions in both the obvious way — dropping the $|\nabla\phi|^2$ and $\nabla\phi \cdot \nabla\eta$ terms — and by allowing the fields to be evaluated at $z = 0$ rather than $z = \eta$. Since $\phi_z(x, y, \eta, t) \simeq \phi_z(x, y, 0, t) + \eta\phi_z z(x, y, 0, t) + ...$, the correction terms from evaluaing at η rather than 0 are indeed quadratic or higher order in the strength of the fields.

With these approximations, the equations become

$$\begin{aligned}
&\nabla^2\phi = 0 \\
&\frac{\partial}{\partial z}\phi = 0 \quad at \quad z = -H \\
&\frac{\partial}{\partial t}\phi = g\eta + p_a/\rho_0 \\
&\frac{\partial}{\partial t}\eta = -\frac{\partial}{\partial z}\phi \quad at \quad z = 0
\end{aligned}$$

DISPERSION RELATION:

For $p_a = 0$ and soutions which are plane waves in the horizontal, we have

$$\eta = \eta_0 \exp(\imath \mathbf{k} \cdot \mathbf{x} - \imath \omega t)$$

so that

$$\phi = \imath \frac{g\eta_0}{\omega} \eta_0 e^{\imath\theta} \quad at` \quad z = 0$$

whic, together with the lower bc. and Laplace's eqn., implies

$$\phi = \imath \frac{g}{\omega} \eta_0 \frac{\cosh(K[z+H])}{\cosh KH} e^{\imath\theta}$$

with $K \equiv |\mathbf{k}|$. The kinematic equation now tells us

$$\frac{\partial}{\partial t}\eta = -\imath \frac{gK}{\omega} \eta_0 \frac{\sinh(K[z+H])}{\cosh KH} e^{\imath\theta}$$

giving the dispersion relationship

$$\omega^2 = gK \tanh(KH)$$

Short wave limit: For short waves, $KH >> 1 \Rightarrow \tanh(KH) = 1$ and

$$\omega \sim \sqrt{gK} \quad , \quad c = \sqrt{\frac{g}{K}} \quad , \quad \mathbf{c}_g = \frac{1}{2}\sqrt{\frac{g}{K}}\frac{\mathbf{k}}{K}$$

The group velocity is half the phase speed.

Long wave limit: For long waves, $KH << 1 \Rightarrow \tanh(KH) = KH$ and

$$\omega \sim \sqrt{gH}K \quad , \quad c = \sqrt{gH} \quad , \quad \mathbf{c}_g = \sqrt{gH}\frac{\mathbf{k}}{K}$$

The group velocity is equal to the phase speed.

Evolution of an initial disturbance

We shall now look at the evolution of an intial compact disturbance

$$\eta(\mathbf{x}, 0)$$

To see exactly what we need to specify, let's reformulate the equations a bit. From the momentum equations

$$\frac{\partial}{\partial t}\mathbf{u} = -\nabla P$$

and the continuity equation

$$\nabla \cdot \mathbf{u} = 0$$

we can see that the pressure also satisfies Laplace's equation

$$\nabla^2 P = 0$$

with an upper boundary condition

$$P = g\eta \quad at \quad z = 0$$

The lower boundary condition arises from

$$0 = \frac{\partial}{\partial t}w = -\frac{\partial}{\partial z}P \quad \Rightarrow \frac{\partial}{\partial z}P = 0 \quad at \quad z = -H$$

The kinematic condition at the upper boundary becomes

$$\frac{\partial}{\partial t}\eta = w \quad \Rightarrow \quad \frac{\partial^2}{\partial t^2}\eta = \frac{\partial}{\partial t}w = -\frac{\partial}{\partial z}P \quad at \quad z = 0$$

If we Fourier-analyze the surface elevation

$$\eta(x, t) = \iint d\mathbf{k}\, \hat{\eta}(\mathbf{k}, t) \exp(\imath\mathbf{k} \cdot \mathbf{x})$$

and the pressure

$$P(\mathbf{x}, z, t) = \iint d\mathbf{k}\, \hat{P}(\mathbf{k}, z, t) \exp(\imath\mathbf{k} \cdot x)$$

with $K = |\mathbf{k}|$, we have

$$\hat{P}(\mathbf{k}, 0, t) g\hat{\eta}(\mathbf{k}, t)$$

and, from the interior equation

$$\nabla^2 P = 0 = \iint d\mathbf{k} \left(\frac{\partial^2}{\partial z^2} - K^2\right) \hat{P}(\mathbf{k}, z, t) \exp(\imath\mathbf{k} \cdot \mathbf{x})$$

Applying the lower boundary condition gives

$$\hat{P}(\mathbf{k}, x, t) = g\hat{\eta}(\mathbf{k}.t)\frac{\cosh K(z+H)}{\cosh KH}$$

and the kinematic condition gives

$$\frac{\partial^2}{\partial t^2}\hat{\eta} = -gK\tanh(KH)\hat{\eta} = -[\Omega(\mathbf{k})]^2\hat{\eta}$$

This equation makes it clear that we need two conditions at $t = 0$, one on η itself and one on $w = \frac{\partial}{\partial t}\eta$. Given these, we can write the general solution

$$\eta(\mathbf{x}, t) = \iint d\mathbf{k}\, \hat{\eta}_+(\mathbf{k})e^{\imath\mathbf{k}\cdot\mathbf{x} - \imath\Omega(\mathbf{k})t} + \hat{\eta}_-(\mathbf{k})e^{\imath\mathbf{k}\cdot\mathbf{x} + \imath\Omega(\mathbf{k})t}$$

with the first representing waves propagating in the positive $\mathbf{k}$ direction and the second representing waves moving in the opposite direction. These are related to the initial conditions by

$$\hat{\eta}(\mathbf{k}, 0) = \hat{\eta}_+ + \hat{\eta}_- \quad , \quad \hat{\eta}_t(\mathbf{k}, 0) = -\imath\Omega(\hat{\eta}_+ - \hat{\eta}_-)$$

Note: we can also write down the radially symmetric solutions in the case with zero initial vertical velocity, we have

$$\eta(r, t) = \int_0^\infty k dk\, a(k) J_0(kr)\cos(\Omega(k)t)$$

One-D case:

We shall look at the one-dimensional case for simplicity. Furthermore, we can look at only the part corresponding to eastward propagation. Thus, we seek an approximation to

$$\eta = \int dk\, \hat{\eta}_+(k)e^{\imath kx - \imath\Omega(k)t}$$

Consider large x and t but with ratio order 1. We can do this by setting $x = Ut$ and take the limit for large t. Then

$$\eta = \int dk\, \hat{\eta}_+(k)e^{\imath t\tilde{\theta}(k)} \quad , \quad \tilde{\theta}(k) = kU - \Omega(k)t$$

The stationary phase method tells us that most of the contribution to the integral comes from the vicinity of k_s where $\tilde{\theta}'(k_s) = 0$. Elsewhere, the phase changes rapidly (for large t) and the integrand oscillates rapidly with zero net contribution. Alternatively we can move into the complex k plane and see that there is a saddle point in the phase at k_s; if we pass through this saddle point at a 45° angle, the argument of the exponential is strongly peaked. Therefore the main contribution comes from near the saddle point $c_g(k_s) = U$

$$\begin{aligned}\eta &\sim \int dk\, \hat{\eta}_+(k_s) \exp\left(\imath t\tilde{}\theta(k_s) + \frac{1}{2}\imath t\tilde{}\theta''(k_s)(k-k_s)^2 + \ldots\right)\\ &\sim \hat{\eta}_+(k_s) e^{\imath t\tilde{}\theta(k_s)} \int dk\, \exp\left(\frac{1}{2}\imath t\tilde{}\theta''(k_s)(k-k_s)^2\right)\end{aligned}$$

Treating the last integral as a probability integral with variance $\sqrt{-1/\imath t\tilde{}\theta''(k_s)}$ gives

$$\eta \sim \hat{\eta}_+(k_s) e^{\imath t\tilde{}\theta(k_s)+\imath\pi/4} \sqrt{\frac{2\pi}{\theta''(k_s)t}}$$

An observer moving at speed U sees waves with wavenumber k_s and frequency $\Omega(k_s)$ where $c_g(k_s) = U$; the wavenumber and frequency do not change for this observer; the amplitude does decrease as $t^{-1/2}$.

On the other hand, an observer at a fixed x corresponds to U decreasing with time and therefore k_s increasing; the wavelength and period get shorter and shorter as time increases. At fixed t, U increases with x, so that the longer waves appear at the front of the disturbance.

Since the details of the dispersion relation did not really enter, the result holds for any type of dispersive waves propagating in one direction; although we do need to watch out for issues such as the sign of θ'' etc. In two dimensions, the waves decay more rapidly $\sim t^{-1}$ (applying a similar asymptotic expansion to the Bessel function solution can show this).

`Demos, Page 6: Dispersion vs H` *<H=20>* *<H=0.02>* *<max value>*

`Demos, Page 6: Single pulse` *<H=0.02>* *<H=0.2>* *<H=1>* *<H=2>* *<H=20>*

`Demos, Page 6: Two-D` *<top view h=10>* *<side view>* *<max value>*

Imhomogeneous media - wave action

We shall consider long waves embedded in a medium with variable depth H and mean flows $\mathbf{u}$ (all vectors, gradient,... are horizontal for this section). The shallow water equations are

$$\begin{aligned}\frac{\partial}{\partial t}\phi &= g\eta + \frac{1}{2}|\nabla\phi|^2\\ \frac{\partial}{\partial t}\eta &= \nabla\cdot[(H+\eta)\nabla\phi]\end{aligned}$$

We'll assume that the background is a large-scale flow $\overline{\mathbf{u}}(\mathbf{X},T)$ and depth field $H(\mathbf{X}) + \overline{\eta}(\mathbf{X},T)$ [which we call $\overline{H}(\mathbf{X},T)$]. These vary only on space and time scales which are long compared to the wave scales and periods ($\mathbf{X} = \epsilon\mathbf{x}$, $T = \epsilon t$). The perturbations satisfy

$$\begin{aligned}D\phi' &= g\eta'\\ D\eta' + \eta'\nabla\cdot\overline{\mathbf{u}} &= \nabla\cdot[\overline{H}\nabla\phi']\end{aligned}$$

with $D \equiv \frac{\partial}{\partial t} - \nabla\overline{\phi} \cdot \nabla = \frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \nabla$. If we now think of the waves as having structure

$$\eta' = \tilde{\eta}(\mathbf{X}, T) \exp\left(\imath \frac{1}{\epsilon} \theta(\mathbf{X}, T)\right)$$

(meaning the real part, of course), an operation such as $D\eta'$ becomes

$$\begin{aligned} D\eta' &= \epsilon \left(\frac{\partial}{\partial T} + \overline{u} \frac{\partial}{\partial X} + \overline{v} \frac{\partial}{\partial Y} \right) \eta' \\ &= e^{\imath\theta/\epsilon} \left[\imath \left(\frac{\partial \theta}{\partial T} + \overline{u} \frac{\partial \theta}{\partial X} + \overline{v} \frac{\partial \theta}{\partial Y} \right) \tilde{\eta} + \epsilon \left(\frac{\partial}{\partial T} + \overline{u} \frac{\partial}{\partial X} + \overline{v} \frac{\partial}{\partial Y} \right) \tilde{\eta} \right] \\ &= e^{\imath\theta/\epsilon} \left[\imath D_0 \tilde{\eta} + \epsilon D_1 \tilde{\eta} \right] \end{aligned}$$

where D_0 is the algebraic quantity

$$D_0 = \frac{\partial \theta}{\partial T} + \overline{u} \frac{\partial \theta}{\partial X} + \overline{v} \frac{\partial \theta}{\partial Y}$$

and D_1 is the operator

$$D_1 = \frac{\partial}{\partial T} + \overline{u} \frac{\partial}{\partial X} + \overline{v} \frac{\partial}{\partial Y}$$

Likewise the graidient operator will pick up two terms, one from the phase and one from the slow variations

$$\frac{\partial}{\partial x} \longrightarrow \imath \frac{\partial \theta}{\partial X} + \epsilon \frac{\partial}{\partial X} \quad , \qquad \nabla \longrightarrow \imath \boldsymbol{\nabla} \theta + \epsilon \boldsymbol{\nabla}$$

Our equations now become

$$\begin{aligned} \imath D_0 \tilde{\phi} + \epsilon D_1 \tilde{\phi} &= g \tilde{\eta} \\ \imath D_0 \tilde{\eta} + \epsilon D_1 \tilde{\eta} + \epsilon \tilde{\eta} \boldsymbol{\nabla} \cdot \overline{\mathbf{u}} &= (\imath \boldsymbol{\nabla} \theta + \epsilon \boldsymbol{\nabla}) \cdot [\overline{H} (\imath \boldsymbol{\nabla} \theta + \epsilon \boldsymbol{\nabla}) \tilde{\phi}] \end{aligned}$$

and we can expand

$$\tilde{\eta} = \eta + \epsilon \eta_1 + \ldots \quad etc.$$

At lowest order, we get the local wave equations

$$\begin{aligned} \imath D_0 \phi &= g \eta \\ \imath D_0 \eta &= -\overline{H} |\boldsymbol{\nabla} \theta|^2 \phi \end{aligned}$$

which gives the dispersion relation

$$D_0^2 = (\omega - \overline{\mathbf{u}} \cdot \boldsymbol{\nabla} \theta)^2 = g \overline{H} |\boldsymbol{\nabla} \theta|^2 \equiv \hat{\omega}^2$$

Here $\hat{\omega}$ is the intrinsic frequency (that for waves in a medium at rest) and the frequency ω has both an advective and a wave contribution

$$\omega = \overline{\mathbf{u}} \cdot \boldsymbol{\nabla}\theta + \hat{\omega}$$

The amplitude is determined by the first order equations

$$\begin{aligned} \imath D_0\phi_1 + D_1\phi &= g\eta_1 \\ \imath D_0\eta_1 + D_1\eta + \eta\boldsymbol{\nabla}\cdot\overline{\mathbf{u}} &= -H|\boldsymbol{\nabla}\theta|^2\phi_1 + \imath\overline{H}\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\phi + \imath\boldsymbol{\nabla}\cdot[\overline{H}\phi\boldsymbol{\nabla}\theta] \end{aligned}$$

Again, we multiply the first equation by $\imath D_0$ and the second by g and add. The η_1 and ϕ_1 terms cancel (using the dispersion relation) and we are left with

$$\imath D_0 D_1\phi + D_1 g\eta + g\eta\boldsymbol{\nabla}\cdot\overline{\mathbf{u}} = \imath g\overline{H}\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\phi + \imath\boldsymbol{\nabla}\cdot[g\overline{H}\phi\boldsymbol{\nabla}\theta]$$

We substitute the lowest order expression $g\eta = \imath D_0\phi$ and divide by $\imath$ to get the equation for the evolution of the amplitude

$$D_0 D_1\phi + D_1(D_0\phi) + D_0\phi\boldsymbol{\nabla}\cdot\overline{\mathbf{u}} = g\overline{H}\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\cdot[g\overline{H}\phi\boldsymbol{\nabla}\theta]$$

or

$$2D_0 D_1\phi - 2g\overline{H}\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\phi + \phi\left[D_1 D_0 + D_0\boldsymbol{\nabla}\cdot\overline{\mathbf{u}} - \boldsymbol{\nabla}\cdot(g\overline{H}\boldsymbol{\nabla}\theta)\right] = 0$$

To put this in terms of the energy,

$$E/g = \frac{1}{2}g\overline{H}|\boldsymbol{\nabla}\theta|^2|\phi|^2 + \frac{1}{2}g|\eta|^2 = g\overline{H}|\boldsymbol{\nabla}\theta|^2|\phi|^2 + \frac{1}{2}g|\eta|^2 = \hat{\omega}^2|\phi|^2$$

we multiply the equation by $\frac{1}{2}\phi^*$ and add the conjugate to get

$$D_0 D_1|\phi|^2 - g\overline{H}\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}|\phi|^2 + |\phi|^2\left[D_1 D_0 + D_0\boldsymbol{\nabla}\cdot\overline{\mathbf{u}} - \boldsymbol{\nabla}\cdot(g\overline{H}\boldsymbol{\nabla}\theta)\right] = 0$$

Combining the first and third terms and the second and fifth gives

$$D_1(\hat{\omega}|\phi|^2) + \nabla\cdot[g\overline{H}\boldsymbol{\nabla}\theta|\phi|^2] + |\phi|^2\hat{\omega}\boldsymbol{\nabla}\cdot\overline{\mathbf{u}} = 0$$

or

$$\frac{\partial}{\partial t}(\hat{\omega}|\phi|^2) + \nabla\cdot[(\overline{\mathbf{u}} + \frac{g\overline{H}}{\hat{\omega}}\boldsymbol{\nabla}\theta)\hat{\omega}|\phi|^2] = 0$$

But from the equation for the frequency, we find the group velocity is

$$\mathbf{c}_g = \overline{\mathbf{u}} + \sqrt{g\overline{H}}\frac{\nabla\theta}{|\nabla\theta|} = \overline{\mathbf{u}} + \frac{g\overline{H}}{\hat{\omega}}\boldsymbol{\nabla}\theta$$

and our equation for the so-called "wave action" $A = \hat{\omega}|\phi|^2 = E/g\hat{\omega}$ becomes

$$\frac{\partial}{\partial t}A + \boldsymbol{\nabla} \cdot (\mathbf{c}_g A) = 0$$

Action changes locally by fluxing in or out at the group velocity. Note that it is the energy divided by the intrinsic frequency which can now be balanced out, not the energy itself:

$$\frac{\partial}{\partial t}\frac{E}{\hat{\omega}} + \boldsymbol{\nabla} \cdot \left(\mathbf{c}_g \frac{E}{\hat{\omega}}\right) = 0$$

EXAMPLES:

For a first example, consider waves travelling into shallow water, $H = H(x)$. We'll start with the waves impinging at an angle with wavenumber $\mathbf{k}_0$. The equations for changes along a ray

$$\begin{aligned}
\frac{\partial \omega}{\partial t} + \mathbf{c}_g \cdot \nabla \omega &= 0 \\
\frac{\partial k}{\partial t} + \mathbf{c}_g \cdot \nabla k &= -\frac{1}{2}\sqrt{\frac{g}{H}}\frac{\partial H}{\partial x}K \\
\frac{\partial \ell}{\partial t} + \mathbf{c}_g \cdot \nabla \ell &= 0
\end{aligned}$$

imply that the frequency and y-wavenumber remain fixed. Therefore, when the wave reached position X, its wavenumber is

$$K = \sqrt{\frac{H_0}{H(X)}}K_0 \quad , \quad k = \sqrt{\frac{H_0}{H(X)}K_0^2 - \ell_0^2}$$

The ray itself satisfies the equation

$$\frac{d\mathbf{X}}{dt} = \sqrt{gH(\mathbf{X})}\frac{\mathbf{k}(\mathbf{X})}{K(\mathbf{X})} = \frac{gH(\mathbf{X})}{\omega}\mathbf{k}(\mathbf{X})$$

from which we conclude that

$$\frac{dY}{dX} = \frac{\ell_0}{k(\mathbf{X})} = \left(\frac{H_0 K_0^2}{H(X)\ell_0^2} - 1\right)^{-1/2}$$

For $H(x) = H_0 \exp(-\gamma x)$, we have

$$Y = \frac{2}{\gamma}\tan^{-1}\left(\sqrt{\frac{K_0^2}{\ell_0^2}e^{\gamma x} - 1}\right)$$

But the essential character is clear from the expressions for the wavenumber and the trajectories: as the depth decreases, the cross-shelf wavenumber increases so that the waves are short and align more parallel with the coast. The trajectory slope decreases, again indicating a turning until the waves are propagating perpendicular to the coast.

Since the intrinisc frequency is just ω, and it's fixed, the energy satisfies

$$\frac{\partial}{\partial t}E + \mathbf{c}_g \cdot \nabla E = -E\nabla \cdot \mathbf{c}_g = -\frac{gE}{\omega}\frac{\partial}{\partial x}(Hk)$$

As the depth gets small, the group velocity behaves as $\sqrt{H}$ so that the fluxes are convergent and the energy density increases.

If we add an along-shore current $v(x)$, the dispersion relation becomes

$$\omega = v\ell + \sqrt{gH(x)}K$$

and ω and ℓ are still invariant along the trajectories. If $v(x)$ exceeds $\sqrt{gH_0}$, we will see reflection of some waves back off-shore. Otherwise if the along-shore velocity reahes some limit as the water shoals, K will still increase as $H^{-1/2}$ so the waves still become parallel to the shore. However, the intrinsic frequency is now $\hat{\omega} = \omega - v\ell$ and decreases as the velocity increases. Thus the increases in wave action associated with the convergence of $\mathbf{c}_g$ will not be entirely reflected in the energy $E = \hat{\omega}A$.

Ship waves