Hough modes

We can reduce the horizontal structure equations

$$\frac{\partial}{\partial t}\mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla P$$
$$\frac{1}{gH_e}\frac{\partial P}{\partial t} + \nabla \cdot \mathbf{u} = 0$$

to a single variable equation by combining $\frac{\partial}{\partial t}$ of the momentum equation with $-f\hat{\mathbf{z}}\times$ to get

$$\frac{\partial^2}{\partial t^2}\mathbf{u} + f^2\mathbf{u} = -\nabla\frac{\partial P}{\partial t} + f\hat{\mathbf{z}} \times \nabla P$$

For single frequency modes, we have

$$\mathbf{u} = -\frac{1}{f^2 - \omega^2}\nabla\frac{\partial P}{\partial t} + \frac{f}{f^2 - \omega^2}\hat{\mathbf{z}} \times \nabla P$$

Substitution into the mass equation (and changing signs) gives

$$\nabla \cdot \frac{1}{f^2 - \omega^2}\nabla\frac{\partial P}{\partial t} - \frac{1}{gH_e}\frac{\partial P}{\partial t} - \hat{\mathbf{z}} \times \nabla P \cdot \nabla\left(\frac{f}{f^2 - \omega^2}\right) = 0$$

But this is not necessarily the most practical approach to computing the modes. Instead, following Longuet-Higgins (1968) let us take the nondimensional equations and find the decomposition into streamfunction and potential

$$\mathbf{u} = -\nabla\phi - \nabla \times \hat{\mathbf{z}}\psi = -\nabla\phi + \hat{\mathbf{z}} \times \nabla\psi$$

so that the horizontal divergence is $\nabla \cdot \mathbf{u} = -\nabla^2\phi$ and the vorticity is $\hat{\mathbf{z}} \cdot \nabla \times \mathbf{u} = \nabla^2\psi$. The vertical vorticity equation becomes

$$\frac{\partial}{\partial t}\zeta + \nabla \cdot (f\mathbf{u}) = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\zeta + f\nabla \cdot \mathbf{u} + \beta v = 0$$

or (scaling time by $1/2\Omega$)

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial\psi}{\partial\lambda} - \sin\theta\nabla^2\phi - \cos\theta\frac{\partial\phi}{\partial\theta} = 0 \tag{1}$$

The divergence equation gives

$$\frac{\partial}{\partial t}\nabla \cdot \mathbf{u} + \nabla \cdot (\hat{\mathbf{z}} \times f\mathbf{u}) = -\nabla^2 P \quad \Rightarrow \frac{\partial}{\partial t}\nabla \cdot \mathbf{u} - f\zeta + \beta u = -\nabla^2 P$$

or (scaling pressure by 2Ω times the velocity)

$$\frac{\partial}{\partial t}\nabla^2\phi + \frac{\partial\phi}{\partial\lambda} + \sin\theta\nabla^2\psi + \cos\theta\frac{\partial\psi}{\partial\theta} = \nabla^2 P \tag{2}$$

Finally, we have

$$\frac{4\Omega^2 a^2}{gH_e}\frac{\partial}{\partial t}P = \nabla^2\phi \tag{3}$$

Barotropic case: When the separation constant $1/gH_e$ is zero, we can find a simpler form: the horizontal velocities are nondivergent so that

$$\mathbf{u} = -\nabla \times \psi \hat{\mathbf{z}} = \hat{\mathbf{z}} \times \nabla \psi$$

and the fully nonlinear vertical vorticity equation becomes

$$\frac{\partial}{\partial t} q + \hat{\mathbf{z}} \cdot (\nabla \psi \times \nabla q) = 0 \quad , \qquad q = \nabla^2 \psi + f$$

In the linear case, this reduces to

$$\frac{\partial}{\partial t} \nabla^2 \psi + \hat{\mathbf{z}} \times \nabla \psi \cdot \nabla f = 0$$

If we look for a form $\psi \sim \exp(\imath m \lambda - \imath \omega t)$ and scale ω by 2Ω, we get the same vorticity equation

$$\frac{\partial}{\partial t} \nabla^2 \psi + \beta v = -\imath \omega \nabla^2 \psi + \cos\theta \frac{\imath m}{\cos\theta} \psi = 0$$

or

$$\frac{1}{\cos\theta} \frac{\partial}{\partial \theta} \cos\theta \frac{\partial}{\partial \theta} \psi - \frac{m^2}{\cos^2\theta} \psi - \frac{m}{\omega} \psi = 0$$

This has Helmholtz solutions

$$\nabla^2 \psi = -K^2 \psi \quad , \qquad \omega = -m/K^2$$

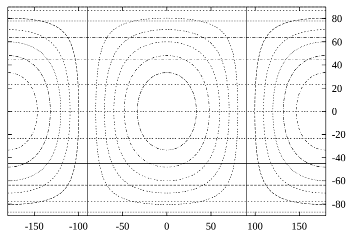
The horizontal structures are Legendre functions

$$\psi = P_n^m(\sin\theta)$$

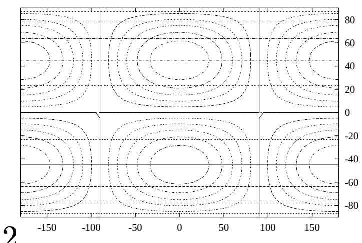
with $K^2 = n(n+1)$ being the total wavenumber. Examples are

$P_n^m(x)$	ψ	ω	ω_{dim}
$P_0^0 = 1$	1	0	0
$P_1^0 = x$	$\sin\theta$	0	0
$P_1^1 = -\sqrt{1-x^2}$	$-\cos\theta\cos\lambda$	$-1/2$	$-\Omega$
$P_2^0 = (3x^2-1)/2$	$(3\sin^2\theta - 1)/2$	0	0
$P_2^1 = -3x\sqrt{1-x^2}$	$-\frac{3}{2}\sin 2\theta \cos\lambda$	$-1/6$	$-\Omega/3$
$P_2^2 = 3(1-x^2)$	$3\cos^2\theta\cos 2\lambda$	$-1/3$	$-2\Omega/3$
$P_3^0 = \frac{1}{2}(5x^2-3)$	$\frac{1}{2}(5\sin^2\theta - 3)$	0	0
$P_3^1 = \frac{3}{2}(1-5x^2)\sqrt{(1-x^2)}$	$\frac{3}{2}(1-5\sin^2\theta)\cos\theta\cos\lambda$	$-1/12$	$-\Omega/6$
$P_3^2 = 15x(1-x^2)$	$15\sin\theta\cos^2\theta\cos 2\lambda$	$-1/6$	$-\Omega/3$
$P_3^3 = -15(1-x^2)^{3/2}$	$-15\cos^3\theta\cos 3\lambda$	$-1/4$	$-\Omega/2$

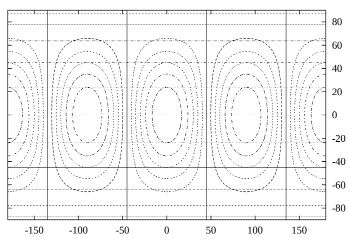
$n = 1,\ m = 1$



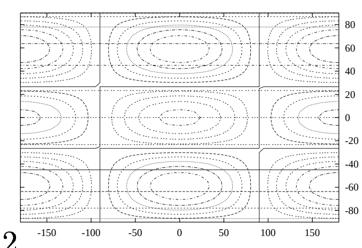
$n = 2,\ m = 1$



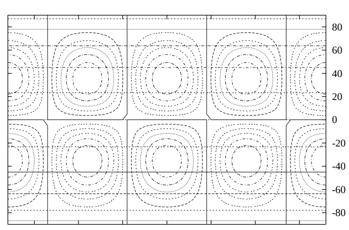
$n = 2,\ m = 2$



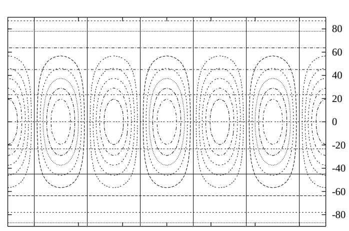
$n = 3,\ m = 1$



$n = 3,\ m = 2$

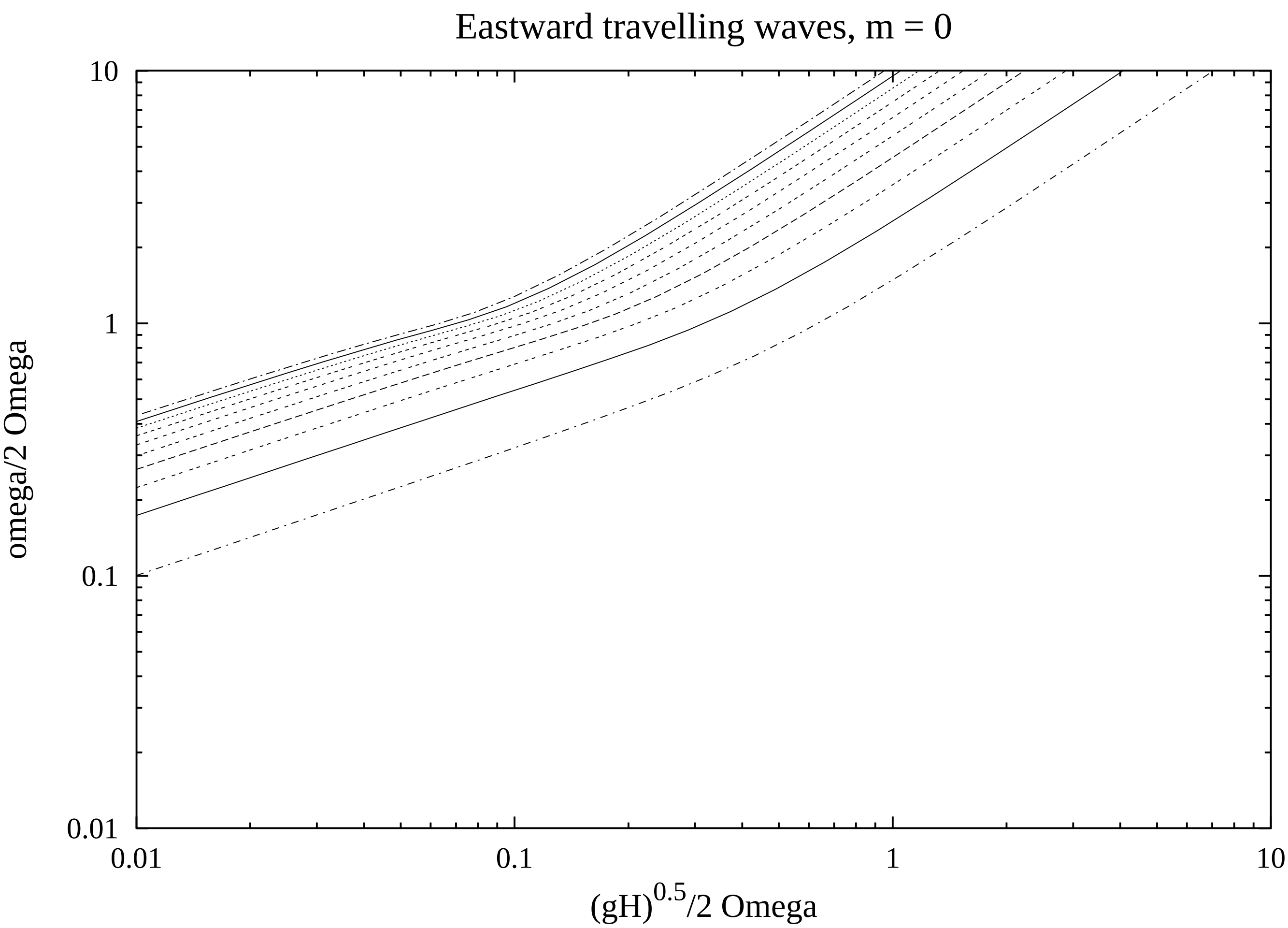


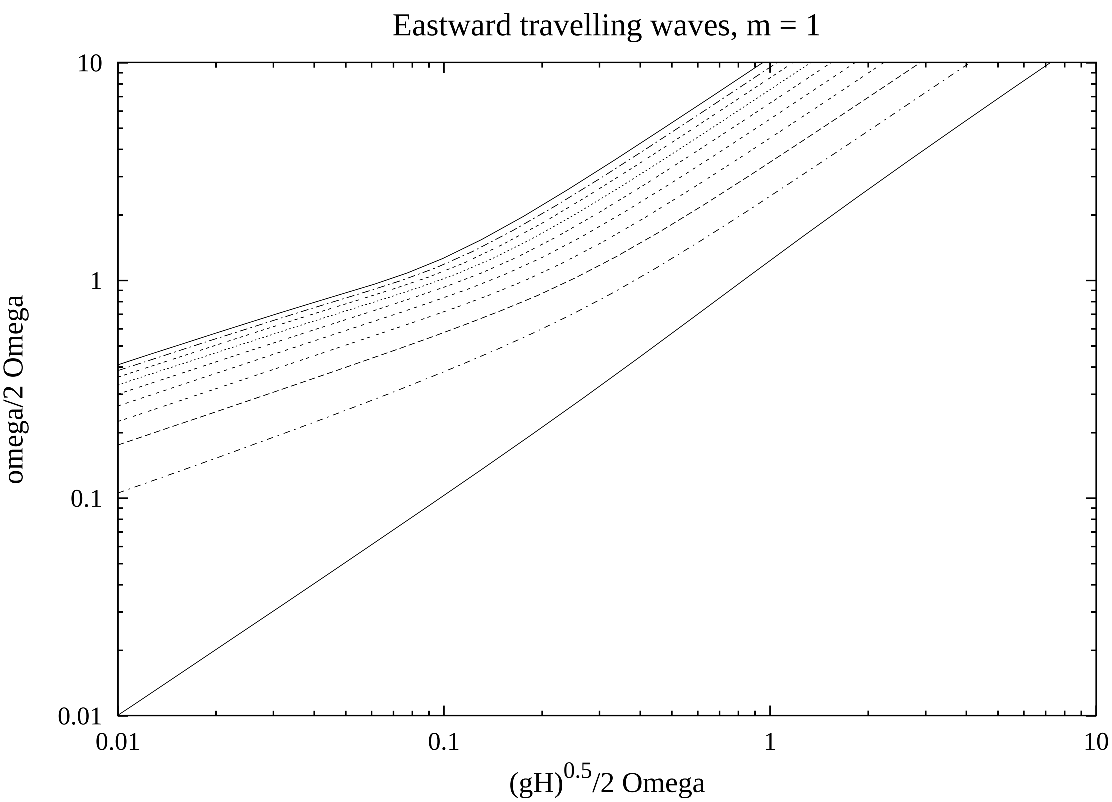
$n = 3,\ m = 3$

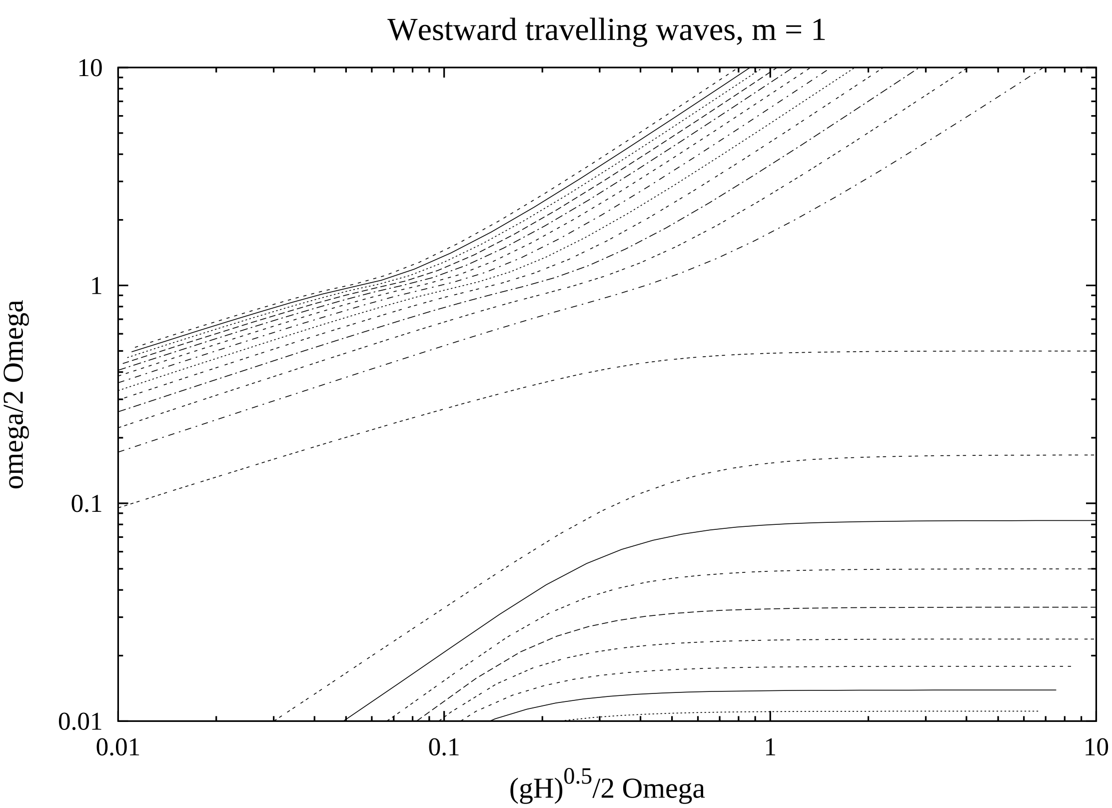


Back to general case...

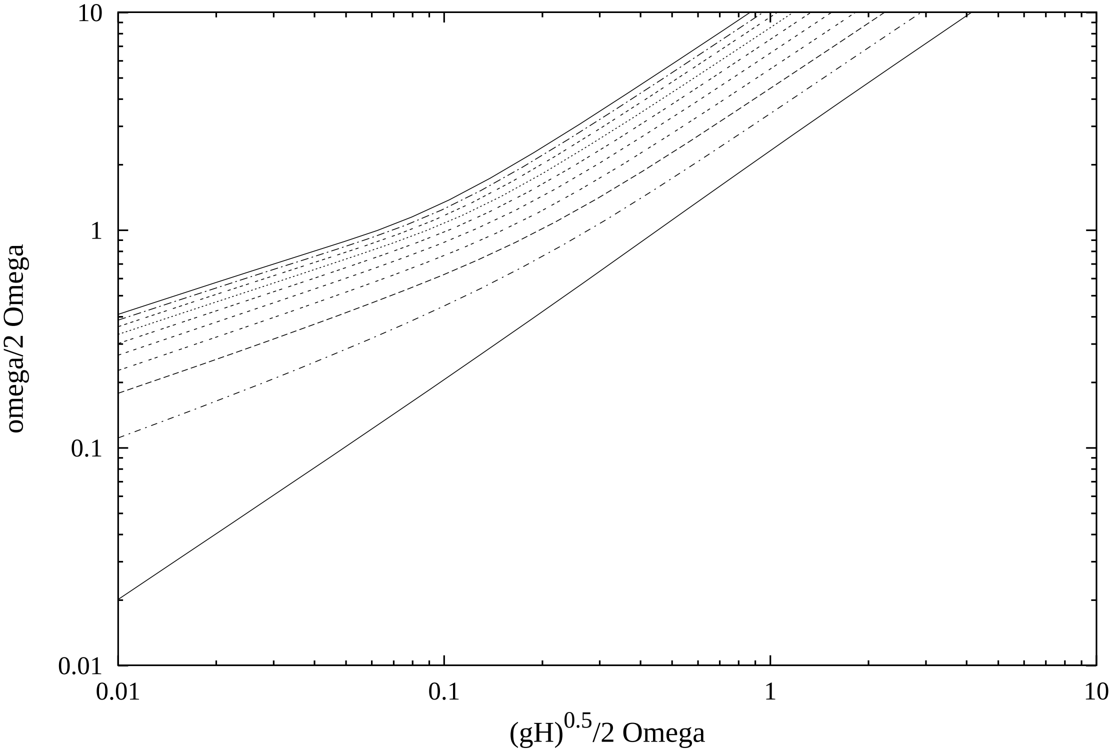
Longuet-Higgins solves by expanding the vorticity and divergence equations in a series of Legendre polynomials; using the recursion relations allows the two to be written as matrix equations for the coefficients and the final problem can be solved in terms of the eigenvalues. The resulting dispersion relation relates the separation constant, m, and the latitudinal structure to the frequency.




Eastward travelling waves, m = 1
10
1
0.1
0.01
omega/2 Omega
0.01
0.1
1
10
(gH)^0.5/2 Omega


Westward travelling waves, m = 1
10
1
0.1
0.01
omega/2 Omega
0.01
0.1
1
10
(gH)^0.5/2 Omega

Eastward travelling waves, m = 2



Westward travelling waves, m = 2