Baroclinic Instability - Eady model

Baroclinic instability generates eddies/ waves from a geostrophically balanced, vertically sheared flow. Because of the thermal wind relationship, the shear flow has horizontal buoyancy gradients, meaning that the height of isentropes varies from place to place. Thus potential energy is stored in the field and can be released by allowing these deviations to relax, lowering some of the dense fluid that's been raised and raising some of the lighter fluid that's been pushed down. We shall analyze this using the Boussinesq, quasi-geostrophic equations.

$$\frac{\partial}{\partial t}\zeta + \mathbf{u}\cdot\nabla(\zeta + \beta y) = f\frac{\partial}{\partial z}w$$

$$\frac{\partial}{\partial t}b + \mathbf{u}\cdot\nabla b + wN^2 = 0$$

$$\mathbf{u} = \left(-\frac{\partial \psi}{\partial y}\ ,\ \frac{\partial \psi}{\partial x}\right) \quad , \quad \zeta = \nabla^2\psi \quad , \quad b = f\frac{\partial}{\partial z}\psi$$

The basic state has a uniformly shear wind profile, $\overline{u} = \Gamma z$ and a corresponding buoyancy gradient $\overline{b} = -\Gamma f y$. We shall also ignore the β effect; in that case the potential vorticity

$$q = \zeta + f\frac{\partial}{\partial z}\left(\frac{b}{N^2}\right)$$

has a constant background value, $\overline{q} = 0$. Therefore the linearized potential vorticity equation becomes

$$(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x})\ q' + v'\frac{\partial \overline{q}}{\partial y} = 0$$

implies $q' = 0$. Thus the dynamics become set by the boundary conditions. When we linearize the buoyancy equation, we get

$$\frac{\partial}{\partial t}b' + \overline{u}\frac{\partial}{\partial x}b' + v'\frac{\partial}{\partial y}\overline{b} + w'N^2 = 0$$

and apply it at a boundary where $w' = 0$, we have

$$\frac{\partial}{\partial t}\frac{\partial \psi'}{\partial z} + \overline{u}\frac{\partial}{\partial x}\frac{\partial \psi'}{\partial z} - \frac{\partial \overline{u}}{\partial z}\frac{\partial}{\partial x}\psi' = 0$$

Eady edge waves

Although it might seem that this system does not support Rossby waves since $\frac{\partial \overline{q}}{\partial y} = 0$, it has an analogue to the Kelvin wave supported by the boundaries. If we consider the lower boundary only, take N^2 to be constant, and look for solutions $\psi' = \Psi(z)\exp(\imath kx)$, we have

$$\left[\frac{f^2}{N^2}\frac{\partial^2}{\partial z^2} - k^2\right]\Psi = 0$$

$$\frac{\partial}{\partial t}\frac{\partial \Psi}{\partial z} - \Gamma\Psi = 0 \quad at \quad z = 0$$

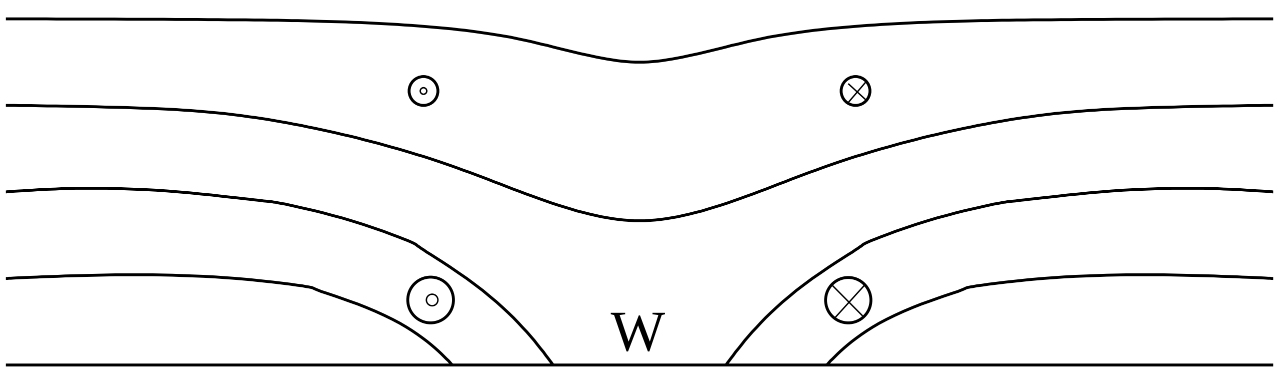
This has a solution

$$\Psi = -\frac{b_0}{kN}e^{-kNz/f}e^{-\imath kct}$$

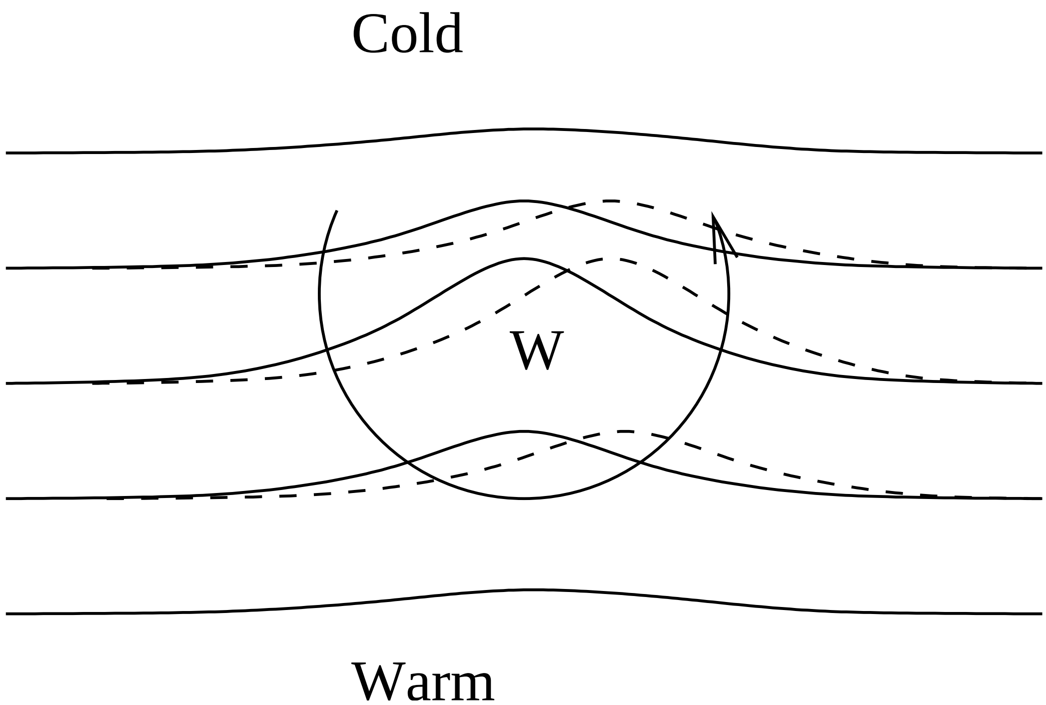
[giving a buoyancy anomaly $b' = b_0\exp(-kNz/f)$] which propagates eastward at

$$c = \Gamma f/kN$$

To understand these waves, consider the following picture of the buoyancy distribution (for which we'll use temperature-like terms since the buoyancy is proportional to the potential temperature in the atmosphere, and is mostly determined by T in the ocean): if we have a warm surface anomaly, the isotherms draw downwards above it. Correspondingly, the thermal wind relationship tells us the wind becomes more anticyclonic going upwards, or, conversely, more cyclonic going from aloft down towards the surface. Since the signal dies out at large z, we can associate high surface temperatures with cyclonic circulation at the surface and cold anomalies with anticyclonic flow.



Associated with the zonal shear is a northward decrease in temperature along the boundary; thus, the cyclonic flow will bring warmer fluid northward, increasing the temperature on the eastern side of the anomaly and colder fluid southward, decreasing the temperature to the west. Therefore the warm anomaly will shift to the east.



Warm perturbations on the upper level, by similar arguments, correspond to anticyclonic flows and will propagate to the west relative to the mean flow.

Instability

Now we consider Eady's (1949) problem with two boundaries, one at $z = 0$ and one at $z = H$. We can solve this a number of ways, but the most illuminating is perhaps to consider changes in the buoyancy anomaly on each boundary. We can write the buoyancy field as

$$b = b_0(t)\frac{\sinh K(H-z)}{\sinh KH} + b_1(t)\frac{\sinh Kz}{\sinh KH}$$

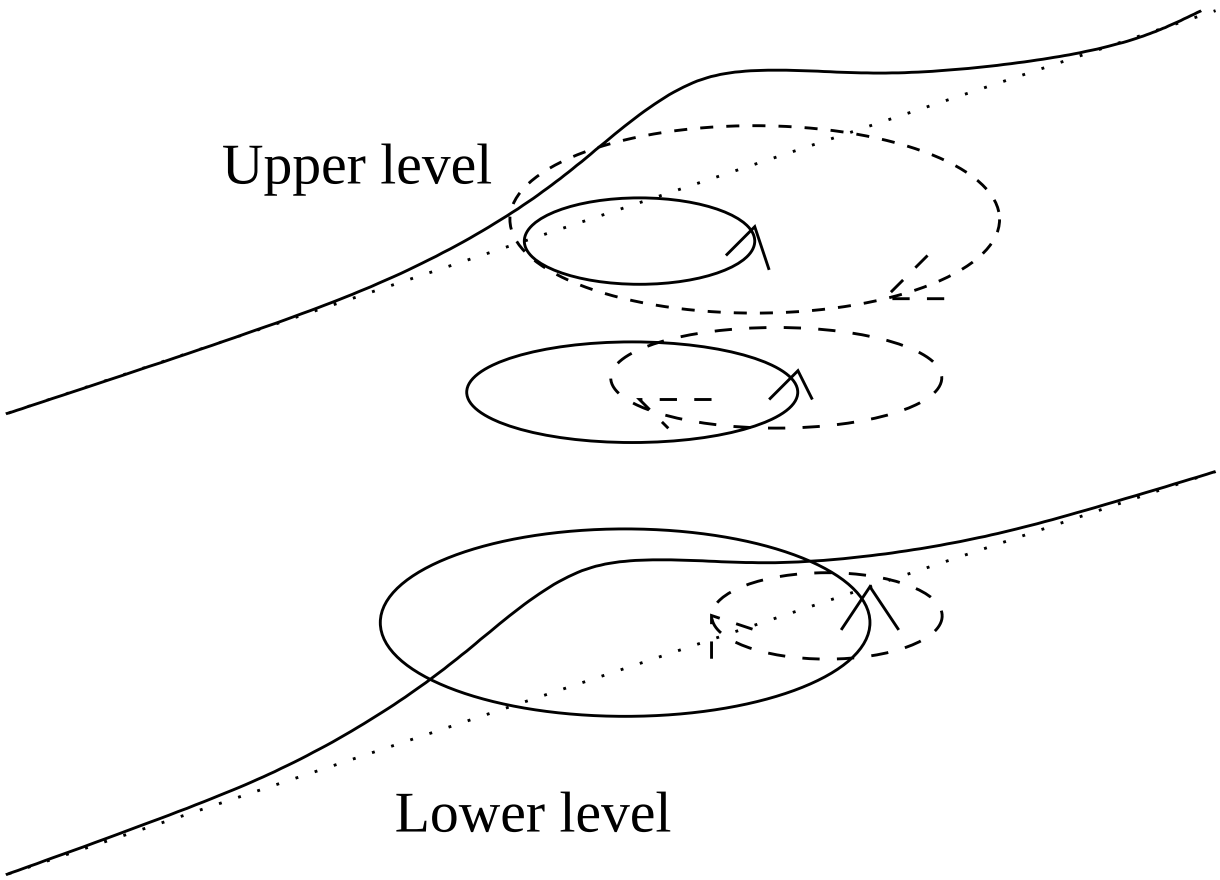
with $K = kN/f$. Integrating, we get

$$\Psi = -b_0(t)\frac{\cosh K(H-z)}{Kf\sinh KH} + b_1(t)\frac{\cosh Kz}{Kf\sinh KH}$$

The equations for the buoyancy evolution become

$$\frac{\partial}{\partial t}b_0 - \imath\frac{f}{N}\Gamma[-b_0\cosh KH + b_1]/\sinh(KH) = 0$$

$$\frac{\partial}{\partial t}b_1 + \imath k\Gamma H b_1 - \imath\frac{f}{N}\Gamma[-b_0 + b_1\cosh KH]/\sinh(KH) = 0$$



If we arrange the anomalies as shown, the flows will reinforce each other, and the perturbations can grow. Mathematically, we have

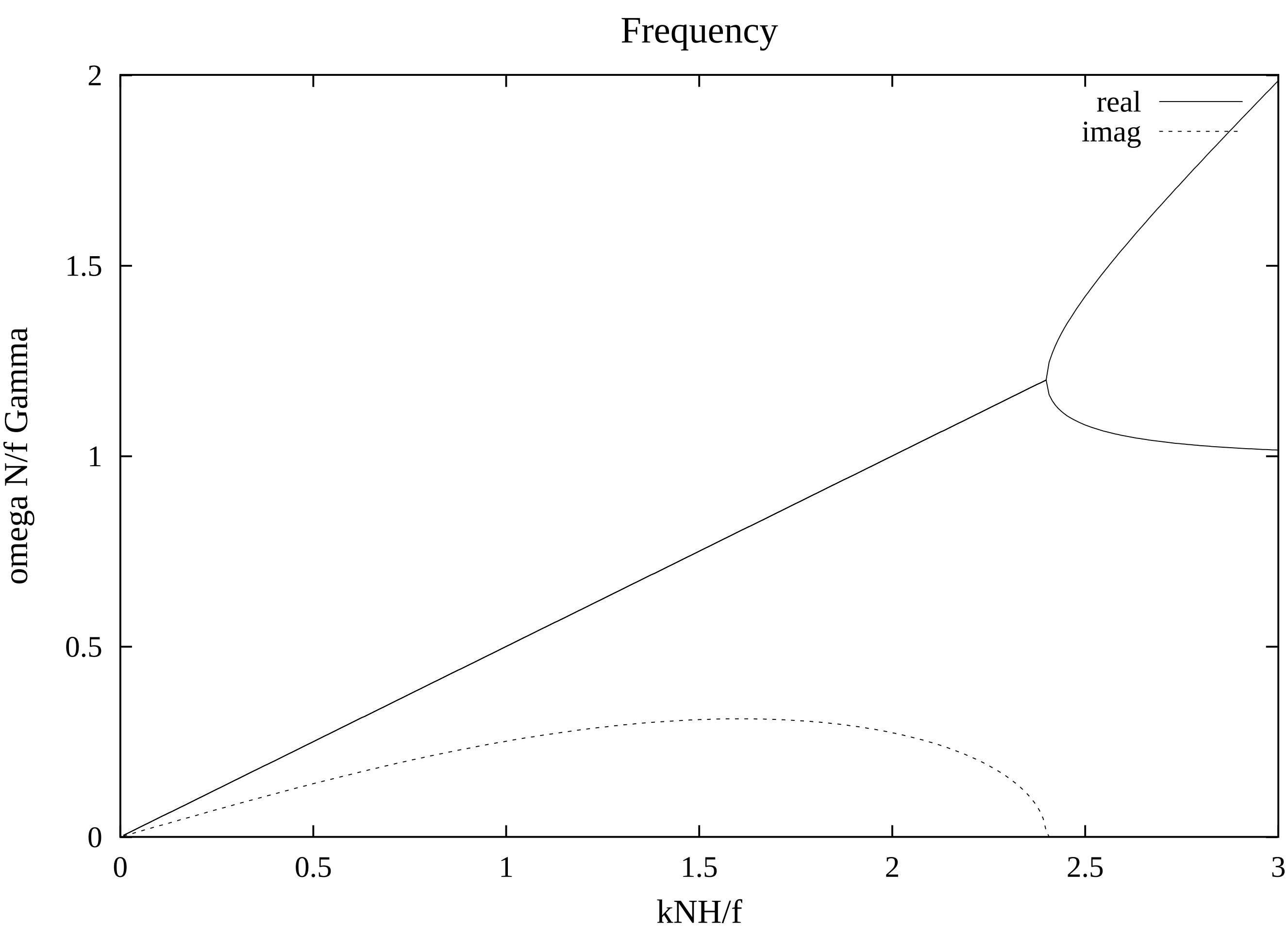
$$\begin{pmatrix} \dfrac{\cosh KH}{\sinh KH} & -\dfrac{1}{\sinh KH} \\ \dfrac{1}{\sinh KH} & KH - \dfrac{\cosh KH}{\sinh KH} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \frac{N\omega}{f\Gamma} \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$$

The flow is unstable for

$$(KH)^2 < 4\left[(KH - \frac{\cosh KH}{\sinh KH})\frac{\cosh KH}{\sinh KH} + \frac{1}{\sinh^2 KH}\right]$$

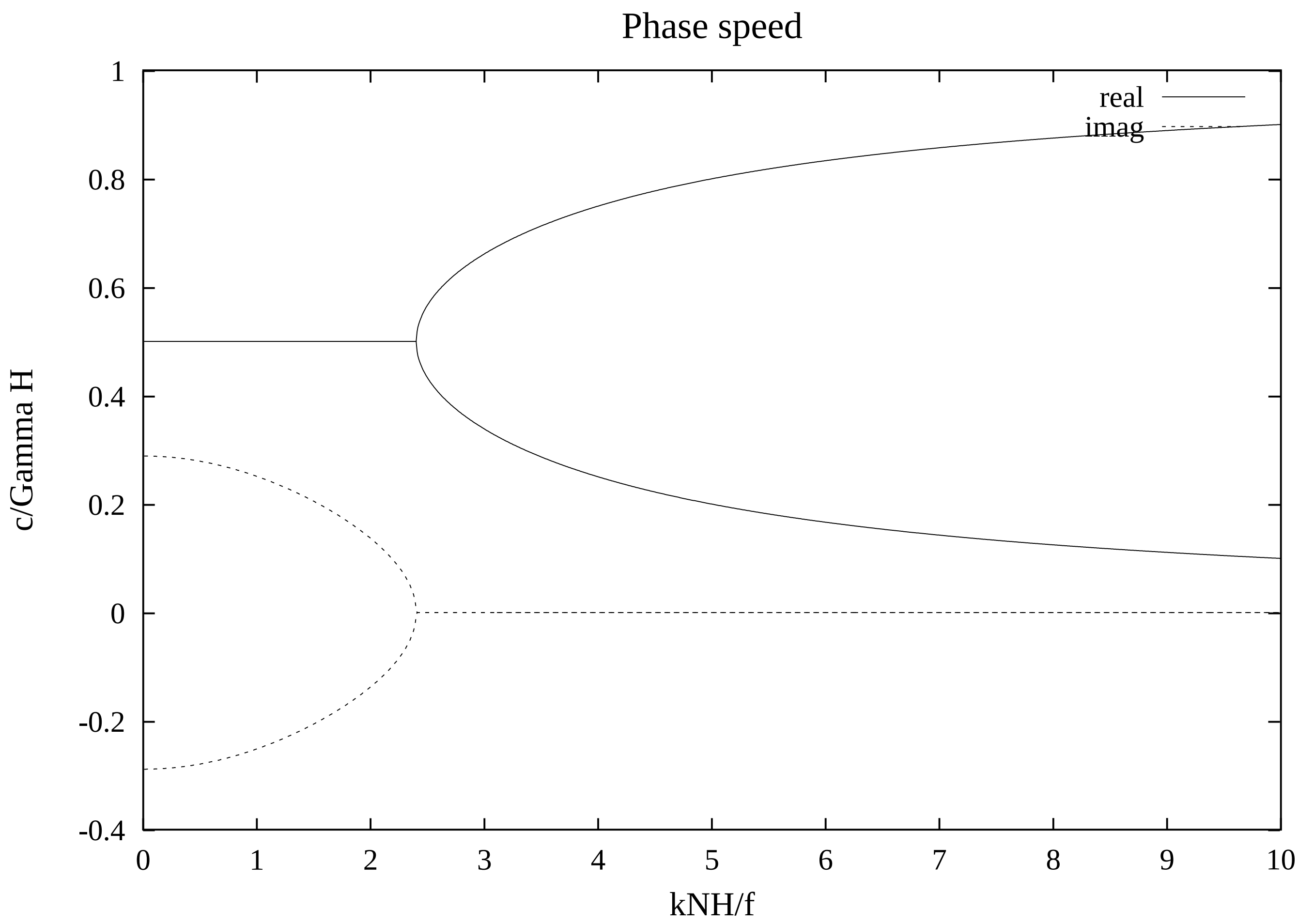
or (multiplying by $\sinh^2 KH$ and factoring)

$$(\cosh KH - \frac{KH}{2}\sinh KH)^2 > 1 \quad \Rightarrow \quad KH \sinh KH < 2 + 2\cosh KH \quad , \quad KH < 2.39936$$



Frequency $N\omega/\Gamma f$ vs. wavenumber KH

The phase speed plot shows the short waves being trapped to the upper or lower boundary, while the long waves can interact and grow.



phase speed $c/\Gamma H$ vs. wavenumber KH

Heat flux

We can calculate the heat flux for $b_0 = \cos kx$, $b_1 = d\cos(kx - \phi)$,

$$\begin{aligned}\overline{v'b'} &= \frac{d\sin\phi}{2\sinh^2(KH)N}[\sinh(Kz)\cosh(Kz-KH) - \cosh(Kz)\sinh(Kz-KH)] \\ &= \frac{d\sin\phi}{2N\sinh(KH)}\end{aligned}$$

Thus for $0 < \phi < \pi$ (as sketched above), we will be fluxing heat northward and (in a bounded system) reducing the temperature gradient.