13. The Kelvin Wave

Let us take the linearized shallow water equation on a f-plane with D = constant.

A solid wall makes possible a rather special wave trapped at the wall, a gravity wave modified by rotation. Consider the following configuration:

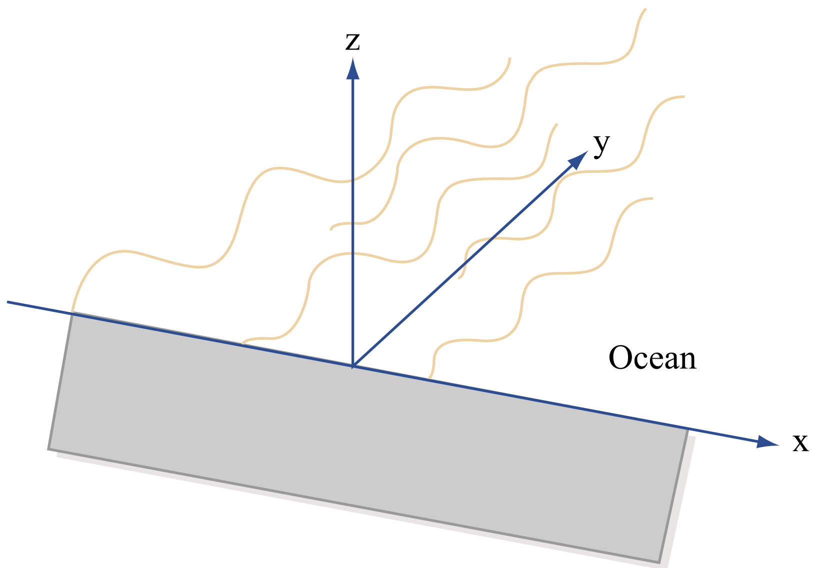

Figure 13-1

Figure by MIT OpenCourseWare.

The peculiar property of the wave is that the velocity normal to the wall is identically zero everywhere

$v \equiv 0$

Then the equations of motion are:

1. $u_t = -g\eta_x$

The pressure gradient along the wall determines the acceleration

2. $fu = -g\eta_y$ — the velocity along the wall is in geostrophic balance with the pressure gradient perpendicular to the wall

3. $\eta_t + Du_x = 0$ — the sea surface adjusts itself to the velocity gradient along the wall

Find one equation for η

From 1 $\quad g\eta_x = -u_t$

From 3 $\quad \eta_t = -Du_x \Rightarrow \eta_{tt} = -D\frac{\partial}{\partial x}(u_t)$

$\frac{\partial}{\partial x}$ of 1 $\quad g\eta_{xx} = -\frac{\partial}{\partial x}(u_t) = \frac{1}{D}\eta_{tt}$

$\eta_{tt} - gD\eta_{xx} = 0$

Assume the usual dependence $e^{-i\omega t}$

$-\omega_2\eta - gD\eta_{xx} = 0 \quad \Rightarrow \eta_{xx} + \frac{\omega^2}{gD}\eta = 0$

If we now assume a <u>wave traveling along the wall</u>

e^{ikx} so the traveling wave is $e^{i(kx-\omega t)}$

we obtain the dispersion relationship

$\omega^2 = (gD)k^2 \qquad \omega = \pm c_o k \qquad c_o = \sqrt{gD}$

This is the shallow water gravity wave dispersion relationship <u>without rotation!!</u>

So, how does rotation affect the wave?

Put $\eta = a(y)\, e^{i(kx-\omega t)}$

Combine the momentum equations of 1 and 2

$\frac{\partial}{\partial x}$ of 2 $\rightarrow f u_t = -g\eta_{yt}$

From 1: $u_t = -g\eta_x$

or $f\eta_x - \eta_{yt} = 0$

With the given shape of η:

$$fka + \omega \frac{da}{dy} = a$$

$$\frac{da}{a} = -\frac{fk}{\omega} dy \qquad \Rightarrow a(y) = \eta_o \exp\left(\frac{-fk}{\omega} y\right)$$

or as $\frac{\omega}{k} = c_o$

$$a(y) = \eta_o e^{-\frac{f}{c_o} y}$$

Full solution

$$\eta = \eta_o \exp - (f/c_o y) \cos(kx - \omega t)$$

Notice that if the wave is on the y>0 side as in the above configuration

$\lim_{y \to +\infty} \eta = \underline{\underline{\text{finite}}} = 0$ $\qquad c_o = +\sqrt{gD} = \frac{\omega}{k} > 0$

and k>0

The wave travels in the +x direction

If the wave is on the y<0 side

$\lim_{y \to -\infty} \eta = \underline{\underline{\text{finite}}} = 0$ $\qquad c_o = -\sqrt{gD} < 0$

and k<0

The wave travels in the – x direction

In either case, in the northern hemisphere f>0. The Kelvin wave always travels with the wall on its right side.

The wave amplitude decreases exponentially away from the wall. The wave is trapped along the wall by rotation.

Rotation does not affect the particle motion and wave propagation; only traps the wave to the coastline.

Synopsis of Kelvin wave

$v \equiv 0$

1. $u_t = -g\eta_x$
2. $fu = -g\eta_y$
3. $\eta_t + Du_x = 0$

$\eta_{xx} + (\omega^2/gD)\eta = 0$ $\qquad \eta = e^{i(kx-\omega t)}$ $\qquad \Rightarrow \omega = \pm c_o k$, $c_o = \sqrt{gD}$

$\eta = a(y)e^{i(kx-\omega t)}$

$a(y) = \eta_o e^{-f\frac{k}{\omega}y}$

$\eta = \eta_o \exp(-f/c_o y)\cos(kx-\omega t)$ The decay scale is

$$\frac{c_o}{f} = \text{Rossby deformation radius}$$

Rotation does not affect the particle motion or wave propagation, only traps the wave at the coast.

The Kelvin wave always travels with the coast on its right side in the Northern Hemisphere

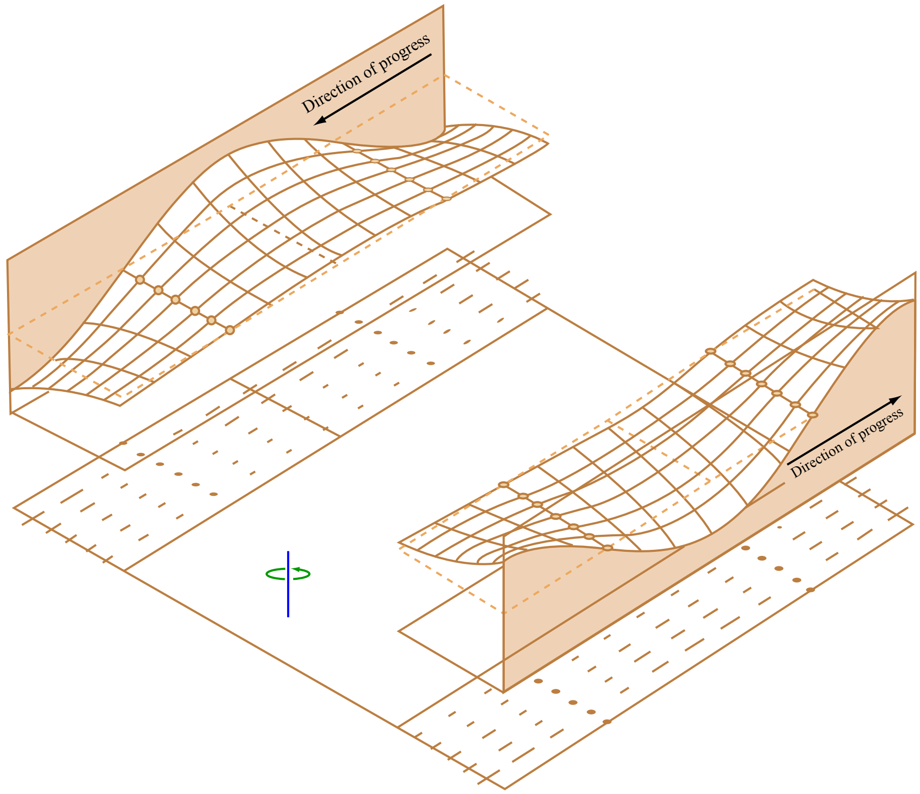

Figure. 13-2 Northern hemisphere Kelvin waves on opposite sides of a channel that is wide compared with the Rossby radius. In each vertical plane parallel to the coast, the currents (shown by arrows) are entirely within the plane and are exactly the same as those for a long gravity wave in a nonrotating channel. However, the surface elevation varies exponentially with distance from the coast in order to give a geostrophic balance. This means Kelvin waves move with the coast on their right in the northern hemisphere and on their left in the southern hemisphere.

Figure by MIT OpenCourseWare.

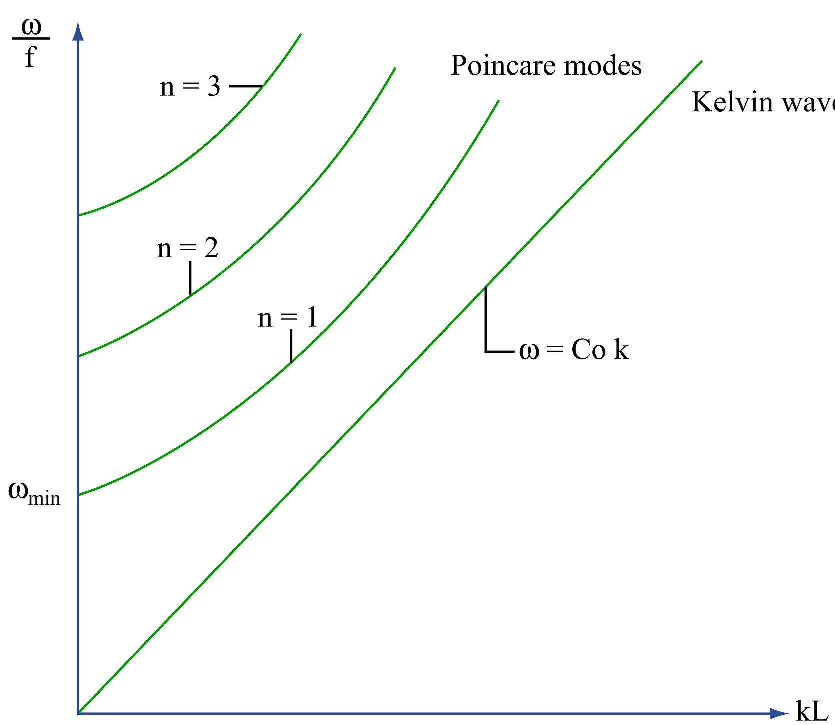

Figure 13-3. Summary dispersion relation for the right moving waves.

Figure by MIT OpenCourseWare.

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