

Lecture 4

4.1 Administration

- Collect problem set 2.
- Hand back quiz 1. Note the following:
 - Remember that Lagrangian and Eulerian solutions will be identical.
 - Material derivative = $D/Dt = (\partial/\partial t + \mathbf{u} \cdot \nabla)$. Boston a common example of advection.
- Distribute problem set 3 (due September 25).
- No class Thursday (September 18).
- The question of “What kind of fluid is a glass?” will be addressed to start the class.
- **Quiz 2!**

4.2 What is a glass?

The question was raised in the previous lecture asking “What kind of fluid is a solid glass?” First, there are many glasses out there (not just the silica glass used in windows). A glass is a material that solidifies in a manner where the individual molecules do not assume a systematic, repeatable structure. That are no crystals (see figure 4.1).

In theory, one could make glass out of almost any material provided you cooled it fast enough that the molecules didn't have time to arrange themselves. A benefit of metal glass would be the ability to mold or cast lightweight, but strong structures. Rather than worry about exotic glasses, I'll focus on regular silica glass.

There is no consensus whether there is a transition temperature at which glass starts flowing, or whether the transition is gradual and the material always flows at sufficiently long times. Because this is a fluids class, I'll vote for the later. This means that silica is a non-Newtonian fluid, but one can never see that it flows!

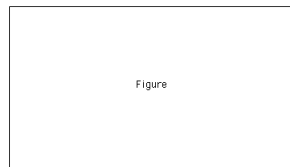


Figure 4.1: (`fig:Glass`) Molecular arrangement of glass is not that of a crystal (left), but rather, there is no systematic or repeatable structure (right).

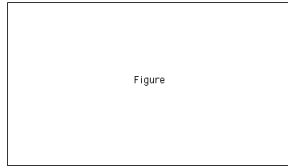


Figure 4.2: (`fig:SteadyFlowStreamStreakTraj`) In steady flow, streaklines, streamlines, and trajectories (solid line) are all identical (the dot is the starting point). The double arrow denotes the direction of the flow.

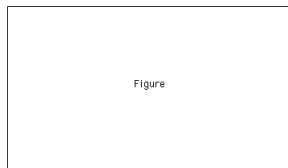


Figure 4.3: (`fig:FlowExample1Flow`) A simple time dependent flow. Double arrows denote direction of flow.

As for “old” glass appearing as though it has flowed, apparently this is an artifact of primitive manufacturing... the glass has always looked that way. The theoretical timescales for the glass to perceptively flow are much longer than a few centuries.

4.3 Trajectories, streaklines, and streamlines

Kinematics is about describing the way flow looks, and three tools for studying this are trajectories, streaklines, and streamlines.

Trajectory: Path taken by a particle.

Streakline: The current location of all particles that pass through the same point.

Streamlines: The curve that is everywhere tangent to instantaneous velocity field.

4.3.1 Steady flow example

In steady flow, streak, stream, and trajectories are all identical (see figure 4.2).

4.3.2 Time dependent flow

Q5: In time-dependent flow, streak lines are identical to streamlines. This is **false**.

As soon as your flow is time dependent, trajectories, streaklines, and streamlines all look very different. Consider the simple example with the flow shown in figure 4.3. The resulting trajectories, streaklines, and streamlines are seen in figures 4.4, 4.5, and 4.6 respectively.

Show film.

4.4 Quantifying viscosity

The aim of the next number of sections is to quantify viscosity.

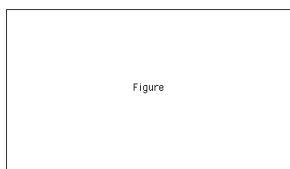


Figure 4.4: (`fig:FlowExample1Trajectory`) Trajectory resulting from simple time dependent flow shown in figure 4.3.

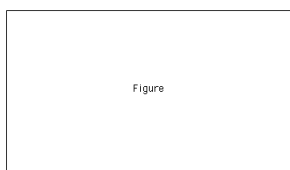


Figure 4.5: (`fig:FlowExample1Streakline`) Streakline resulting from simple time dependent flow shown in figure 4.3.

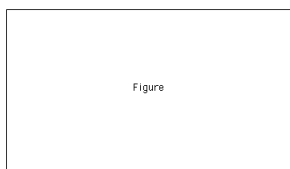


Figure 4.6: (`fig:FlowExample1Streamlines`) Streamlines resulting from simple time dependent flow shown in figure 4.3.

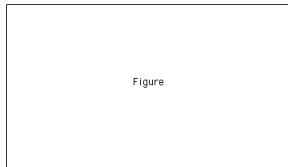


Figure 4.7: (`fig:AxisEvolution`) Marking a fluid blob allows one to see how it is deformed over time. There will be translation, rotation, and deformation.

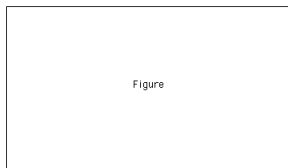


Figure 4.8: (`fig:LinearStrainRate`) Schematic of the application of a linear stress in the x direction.

Kinematics is all about describing the appearance of a fluid, so a sensible question to ask is: “If I have a blob of fluid at time t , how is it likely to be deformed over some time Δt ?” If you mark the fluid with a little orthogonal axis, you can observe how the axis changes as the blob evolves. The result is that you will see translation, rotation, and deformation (see figure 4.7).

This is all formalized in the Cauchy-Stokes decomposition theorem (I’ve seen it expressed as Helmholtz’s 1st law as well). It states:

An arbitrary instantaneous fluid motion may be resolved at each point into a translation, a dialation along three perpendicular axes, and a rigid rotation of those axes.

We’re going to spend some time quantifying this statement. We are interested in kinematics, and want to know how things deform, but the real motivation behind the next couple of lectures will be providing a foundation that will enable us to quantify the viscosity terms in the equations of motion.

Last class I provided a simple relationship between an applied stress and a rate of deformation:

$$\tau_{31} = \mu \dot{\gamma} \tag{4.1}$$

This relationship between an applied stress, τ_{31} , and the resulting fluid deformation, $\dot{\gamma}$, is called a constitutive relationship. It relates stress τ , to strain rate, $\dot{\gamma}$.

This is one way in which a fluid blob can change. I will recast this in multiple dimensions and bring in two more. Thus, we will look at

- Linear strain rate
- Shear strain rate
- Rigid rotation rate

4.5 Linear, or normal, strain rate

Q1: Linear strain rate is defined to be the rate of change in length per length.

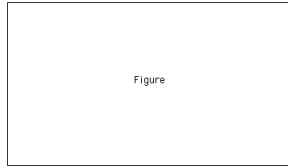


Figure 4.9: (fig:ShearStrainRateDir) Schematic of the application of a shear stress in the x direction.

The linear strain (rate of change in length per length) in the x direction, e_{11} (sometimes \dot{e}_{11}), as shown in figure 4.8 is formulated as follows:

$$\begin{aligned}
 e_{11} \text{ (sometimes } \dot{e}_{11}\text{)} &= \frac{\text{rate of change of length}}{\text{length}} \\
 &= \frac{\frac{\Delta x_2 - \Delta x_1}{\Delta t}}{\Delta x_1} \\
 &= \frac{1}{\Delta t \Delta x_1} (\Delta x_2 - \Delta x_1) \\
 &= \frac{1}{\Delta t \Delta x_1} (\Delta x_1 + (u + \Delta u)\Delta t - u\Delta t - \Delta x_1) \quad (4.2)
 \end{aligned}$$

Now, note that

$$\begin{aligned}
 u = f(x) \Rightarrow u + \Delta u &= f(x + \Delta x) \\
 &= f(x) + \Delta x f'(x) \\
 &= u + \Delta x \frac{\partial u}{\partial x} \quad \text{where } \Delta x = \Delta x_1 \quad (4.3)
 \end{aligned}$$

Note that these last couple steps could have been done by inspection under the existence of a horizontal shear (**Jim: Perhaps this last line merits a bit more commenting**). Thus, by substituting equation 4.3 into 4.2 we get

$$\begin{aligned}
 e_{11} &= \frac{1}{\Delta t \Delta x_1} (\Delta x_1 + (u + \Delta x_1 \frac{\partial u}{\partial x})\Delta t - u\Delta t - \Delta x_1) \\
 &= \frac{1}{\Delta t \Delta x_1} (\Delta x_1 \frac{\partial u}{\partial x} \Delta t) \\
 &= \frac{\partial u}{\partial x} \quad (4.4)
 \end{aligned}$$

Similarly, we find that $e_{22} = \partial v / \partial y$ and $e_{33} = \partial w / \partial z$. Thus, we have quantified the linear strain rate on a fluid. (**Jim: How to best word this last line?**).

4.6 Shear strain rate

The linear strain rate says something about a fluid getting squashed or stretched. Shear strain rate says something about the differential rotation of the axes of a fluid blob. The shear strain rate, $\dot{\gamma} \equiv$ rate of the decrease of the angle formed by two perpendicular lines. It says something about how the angles deform. To describe this, a picture similar to that used to define a fluid is drawn in figure 4.9. Here, shear is considered in only the x direction.

Noting that $\Delta\gamma$ is positive and that $\Delta x = (u + \Delta u)\Delta t - u\Delta t = \Delta u\Delta t$, we have

$$\tan \Delta\gamma = \frac{\Delta x}{\delta y}$$

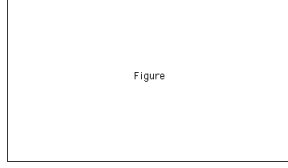


Figure 4.10: (fig:ShearStrainRateydir) Schematic of the application of a shear stress in the y direction.

$$\begin{aligned}
 &= \frac{\Delta u \Delta t}{\delta y} \\
 \Rightarrow \Delta \gamma &= \frac{\Delta u \Delta t}{\delta y} \quad (\text{small angle}) \\
 \Rightarrow \frac{\Delta \gamma}{\Delta t} &= \frac{\Delta u}{\delta y}
 \end{aligned}$$

Where, in the limit as $\Delta t \rightarrow 0$

$$\dot{\gamma} = \frac{\partial u}{\partial y} \quad (4.5)$$

Now we will do the same in the y -direction (see figure 4.10). Here we note that $\Delta \alpha$ is positive and that $\Delta y = (v + \Delta v)\Delta t - v\Delta t = \Delta v\Delta t$.

$$\begin{aligned}
 \tan \Delta \alpha &= \frac{\Delta y}{\delta x} \\
 &= \frac{\Delta v \Delta t}{\delta x} \\
 \Rightarrow \Delta \alpha &= \frac{\Delta v \Delta t}{\delta x} \quad (\text{small angle}) \\
 \Rightarrow \frac{\Delta \alpha}{\Delta t} &= \frac{\Delta v}{\delta x}
 \end{aligned}$$

Where, in the limit as $\Delta t \rightarrow 0$

$$\dot{\alpha} = \frac{\partial v}{\partial x} \quad (4.6)$$

This we now have the rate of angular deformation for each axis. How are they combined to give the shear rate? The average is taken:

$$e_{ij} = \frac{1}{2}(\dot{\gamma} + \dot{\alpha}) = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad (4.7)$$

This gives the contribution of the change of the axis due to shear. Note that this is not all of the axis change – Rigid rotation also contributes.

Q2: Why the average? We will come back to this.

The shear strain rate describes the deformation due to shear, and one of its impacts is a rotation. There is also a contribution to the change of the blob due to the intrinsic rotation (the vorticity) of the fluid.

4.7 Rigid rotation rate

Here we are talking about the component of the axes change due to the rigid rotation (see figure 4.11). The rotation rate is given by the average rotation rate of two initially perpendicular lines.

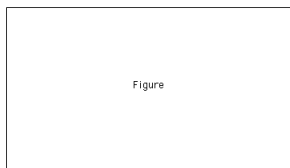


Figure 4.11: (`fig:RigidRotationRate`) Schematic of rigid rotation.

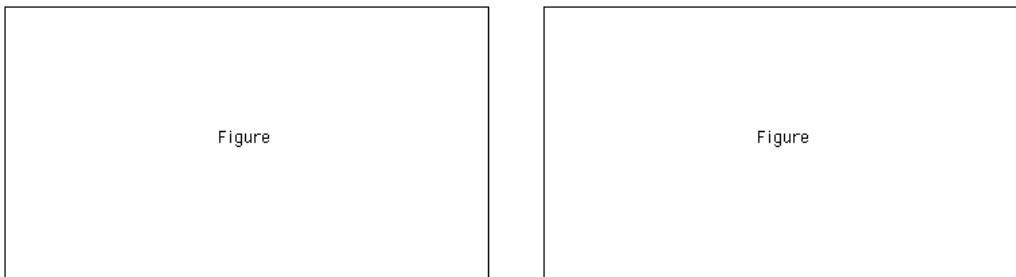


Figure 4.12: (`fig:Gamma`, `fig:Alpha`) Gamma (left panel) and alpha (right panel) geometries.

Given our convention for positive γ and positive α , a positive γ change (by the right hand rule) is a negative angular velocity, and a positive α change is a positive angular velocity. Thus, the respective angular velocities are given by $-\partial\gamma/\partial t$ and $+\partial\alpha/\partial t$. Thus, the average is

$$\frac{1}{2} \left(\frac{\partial\alpha}{\partial t} - \frac{\partial\gamma}{\partial t} \right) \quad (4.8)$$

Now we need to find the relations for $\partial\gamma/\partial t$ and $\partial\alpha/\partial t$ (see figure 4.12). This will proceed exactly like with the shear strain rate derivations. First gamma, then alpha:

$$\begin{aligned} \tan \Delta\gamma &= \frac{\Delta x}{\delta y} = \frac{\Delta u \Delta t}{\delta y} \\ \Rightarrow \frac{\Delta\gamma}{\Delta t} &= \frac{\Delta u}{\delta y} \quad (\text{small angle}) \\ \Rightarrow \frac{\partial\gamma}{\partial t} &= \frac{\partial u}{\partial y} \quad (\text{limit } \Delta t \rightarrow 0) \end{aligned} \quad (4.9)$$

$$\begin{aligned} \tan \Delta\alpha &= \frac{\Delta y}{\delta x} = \frac{\Delta v \Delta t}{\delta x} \\ \Rightarrow \frac{\Delta\alpha}{\Delta t} &= \frac{\Delta v}{\delta x} \quad (\text{small angle}) \\ \Rightarrow \frac{\partial\alpha}{\partial t} &= \frac{\partial v}{\partial x} \quad (\text{limit } \Delta t \rightarrow 0) \end{aligned} \quad (4.10)$$

Thus, the rotation rate, Ω , is

$$\Omega = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \quad (4.11)$$

Note that the vorticity, ω , is defined as $2\Omega = \omega \equiv$ vorticity. Recall, $\omega = \nabla \times \mathbf{u}$.

Q3: Vorticity, ω , is defined as $\omega = 2\Omega =$ twice the rotation rate.

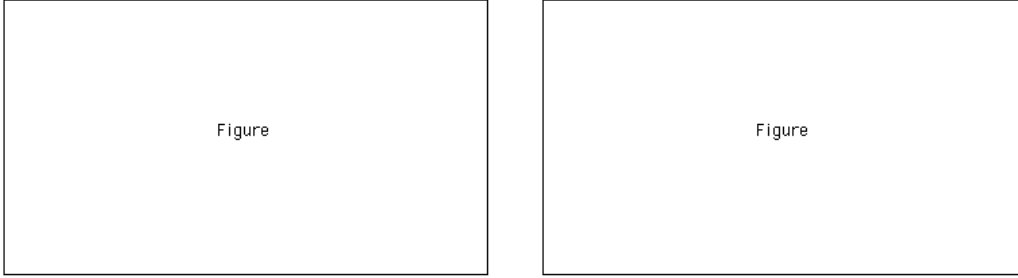


Figure 4.13: (`fig:RigidRotationRate`, `fig:AxisEvolveNonOrthogonal`) The deformation of an axis (left panel) can be thought of as the deformation of a circle (right panel).

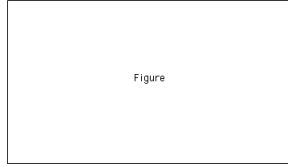


Figure 4.14: (`fig:AxisEvolveOrthogonal1`) Chose a new set of axes such that the original (left) become the axes of the deformed circle (now ellipse), but also the major axes of the ellipse (right).

4.8 Shear strain rate versus rigid rotation rate

Let's talk a little more about the difference between the shear strain rate and the rigid rotation rate. Note figure 4.13. The deformation of an axis (left panel) can be thought of as the deformation of a circle. Some of the change in the axes is due to angular deformation, and some is due to rigid rotation (as we've already seen).

But what if I chose different axes? What if I chose axes that, when operated on by the flow, evolve into the major axes of the ellipse? (See figure 4.14). As it turns out, the axis that evolves into the primary axes of the ellipse start out perpendicular and end up perpendicular (more about this neat fact later). Lets do a coordinate transformation so that my initial axis are horizontal and vertical (figure 4.15).

Thus,

$$\begin{aligned}\tan \Delta\gamma &= \frac{\Delta x'}{\delta y'} \Rightarrow \frac{\partial\gamma}{\partial t} = \frac{\partial u'}{\partial y'} > 0 \text{ here} \\ \tan \Delta\alpha &= \frac{\Delta y'}{\delta x'} \Rightarrow \frac{\partial\alpha}{\partial t} = \frac{\partial v'}{\partial x'} > 0 \text{ here}\end{aligned}$$

Because final-time axes are perpendicular

$$\left| \frac{\partial v'}{\partial x'} \right| = \left| \frac{\partial u'}{\partial y'} \right|, \quad \frac{\partial v'}{\partial x'} = -\frac{\partial u'}{\partial y'}$$

Thus the shear strain rate becomes

$$\frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2} \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \right) = 0$$

And the rigid rotation rate is

$$\frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \frac{1}{2} \left(-\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \right) = -\frac{\partial u}{\partial y}$$

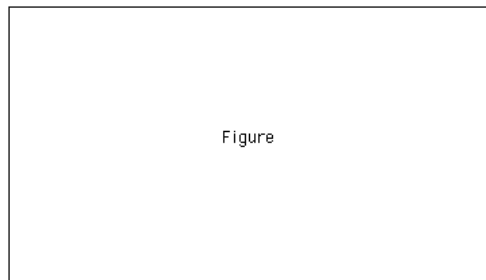
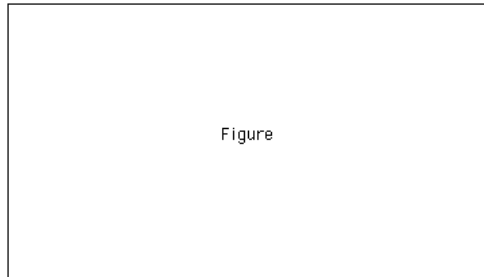


Figure 4.15: (`fig:AxisEvolveOrthogonal2`, `fig:AxisEvolveCoordinateTransformation`)
Quantifying the coordinate transformation.

Thus by transforming my coordinate axis, I can remove all of the mathematical impact of shear strain! Note also that one can now see why the $1/2$ was introduced – so that that the rigid rotation rate does not have a factor of 2 in from of it.

Jim: There is a half page of the notes at the end which you have crossed out and wrote “Note sure” on the side of. It it not included here.

4.9 Reading for class 5

KC01: None assigned in lecture notes.