Lecture 5: Changing Coordinate Systems and Mohr's Circle

Lecture 2 explained that temperature is a zeroth-order tensor, force is a first-order tensor, and stress is a second-order tensor. The order of a tensor is called its rank and is defined by its law of transformation under a change of coordinates. This lecture explains the transformation laws for first- and second-order tensors, and uses these laws to derive a convenient representation of stress called Mohr's circle.

1. Transforming Tensors into Different Coordinate Systems

Changing coordinate systems

In order to transform a tensor into a different coordinate system, one must first understand how to transform the coordinate system itself. Consider the following figure:

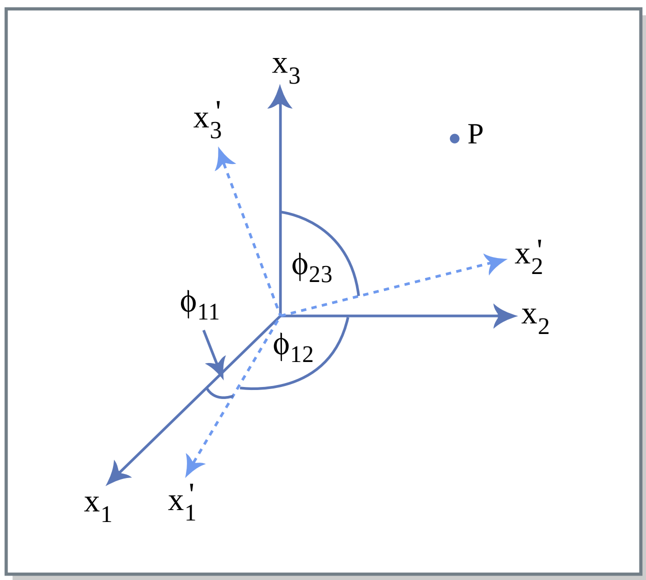

Figure 5.1

Figure by MIT OCW.

If $\hat{x}_i'$ and $\hat{x}_j$ represent unit vectors that are the axes of two coordinate systems with the same origin, they are related by the equation

$$\hat{x}_i' = \alpha_{ij}\hat{x}_j$$

where α_{ij} is the cosine of the angle between the primed axis $\hat{x}_i'$ and the unprimed axis $\hat{x}_j$. For example, α_{12} is the cosine of the angle between $\hat{x}_1'$ and $\hat{x}_2$. α_{ji} represents a 9-component matrix called the transformation matrix. Unlike the stress tensor, it is not symmetric ($\alpha_{ij} \neq \alpha_{ji}$).

$$\alpha_{ij} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

In matix equations, the transformation law is written

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The inverse transformation law is written

$$\hat{x}_i = \alpha_{ji} \hat{x}_j'$$

Example of coordinate transformations

Consider the following transformation of coordinates:

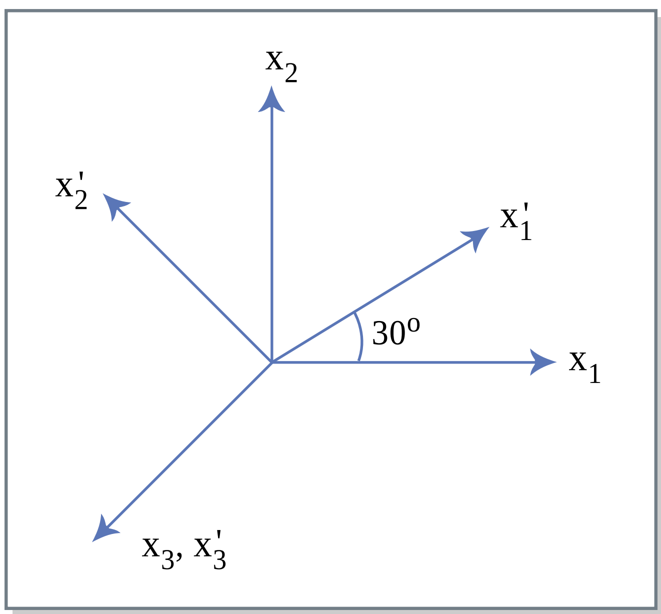

Figure 5.2

Figure by MIT OCW.

The transformation matrix is

$$\alpha_{ij} = \begin{bmatrix} 30° & 60° & 0 \\ 120° & 30° & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The explicit transformation equations are

$$\hat{x}_1' = \cos 30° \hat{x}_1 + \cos 60° \hat{x}_2$$
$$\hat{x}_2' = \cos 120° \hat{x}_1 + \cos 30° \hat{x}_2$$

Since $\hat{x}_i'$ and $\hat{x}_j$ are both unit length, these equations are easy to verify from the picture.

First-order tensors

First-order tensors or vectors have two components in 2D coordinates and three components in 3D coordinates. They transform according to the same laws as coordinate axes because coordinate axes are themselves vectors.

If u_j is a vector in the $\hat{x}_j$ coordinate system and u_i' is a vector in the $\hat{x}_i'$ coordinate system, then the following equations describe their transformation:

$$u_i' = \alpha_{ij} u_j$$
$$u_i = \alpha_{ji} u_j'$$

Note that α_{ij} is positive if the angle is measured counterclockwise from $\hat{x}_i'$ to $\hat{x}_j$. It is negative if the angle is measured clockwise.

Second-order tensors

The transformation law for second-order tensors like stress and strain is more complicated than the transformation law for first-order tensors. It may be derived as follows:

a. Begin with the vector transformation of traction T_i to T_k':

$$T_k' = \alpha_{ki} T_i$$

b. Rewrite T_k' and T_i using Cauchy's formulas:

$$T_k' = \sigma'_{kl} n_l' \text{ and } T_i = \sigma_{ij} n_j$$

Substitute Cauchy's formulas into the original transformation equation:

$$\sigma'_{kl} n_l' = \alpha_{ki} \sigma_{ij} n_j$$

c. Transform the normal vector n_j to n_l' and substitute into the previous equation:

$$n_j = \alpha_{lj} n_l'$$
$$\sigma'_{kl}\, n_l' = \alpha_{ki}\sigma_{ij}\alpha_{lj} n_l'$$

d. Cancel the n_l' term on each side and group the αs:

$$\sigma'_{kl} = \alpha_{ki}\alpha_{jl}\sigma_{ij}$$

Note that changing the position of the last α term changes the order of its subscripts.

In vector notation, the equation is

$$\underline{\underline{\sigma}}' = \alpha \underline{\underline{\sigma}} \alpha^T$$

where the double underbars denote second-rank tensors and the superscript T denotes the transpose of matrix α.

2. Mohr's Circle

Motivation

Lecture II explained that an object resting on a slope will slide down when the shear traction on the slope is greater than or equal to the product of the normal traction and the coefficient of friction.

$$\tau = f_s \sigma_n$$

On a shallow slope, σ_n is large and the object will not slide. On a steep slow, τ is large and the object will slide. For any plane with normal $\hat{n}$, we can calculate if the plane will fail if the stress tensor σ_{ij} at the interface between the object and the slope is known. Calculate σ_n and , τ as follows:

Vector and Tensor Notation	Summation Notation
$\vec{T} = \underline{\underline{\sigma}}\hat{n}$	$T_i = \sigma_{ij} n_j$
$\sigma_n = \vec{T}\cdot\hat{n}$	$\sigma_n = T_i n_i$
$\tau = \vec{T} - \vec{T}\cdot\hat{n}$	$\tau = T_i - T_i n_i$

This method is straightforward but cumbersome. A different approach involves rotating the coordinate system such that x_1' is along $\hat{n}$. In this case σ_n and τ are much easier to derive:

$$\sigma_n = \sigma'_{11}$$
$$\tau = \sigma'_{12}$$

Deriving Mohr's Circle

Mohr's circle may be derived in two or three dimensions. This lecture explains the derivation in two dimensions because it is more straightforward and the results are easier to graph and understand. The derivation assumes that x_1, x_2, and x_3 are principle directions.

Consider the following figure in which the x_i coordinate system is rotated clockwise about the x_3 axis to x_i':

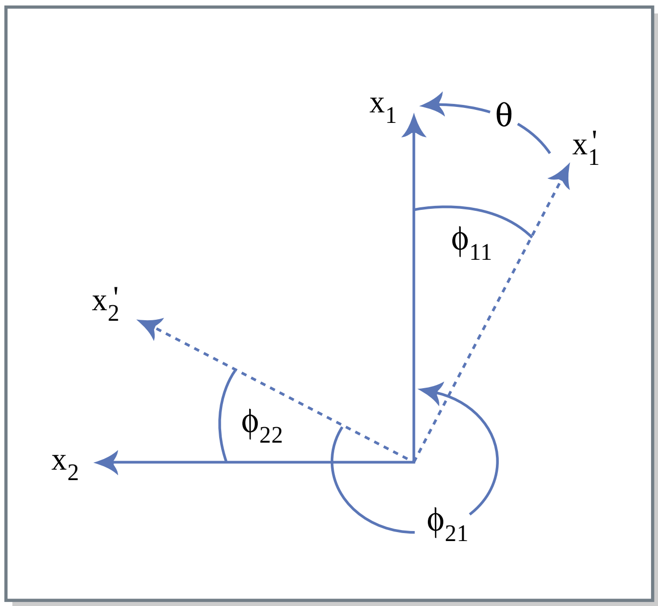

Figure 5.3

Figure by MIT OCW.

The rotation matrix is:

$$\alpha_{ij} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The stress tensor $\underline{\underline{\sigma}}$ in the x_i coordinate system is transformed to $\underline{\underline{\sigma}}'$ in the x_i' coordinate system by the following equation:

$$\underline{\underline{\sigma}}' = \alpha \underline{\underline{\sigma}} \alpha^T$$

$$\underline{\underline{\sigma}}' = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{\sigma}}' = \begin{bmatrix} \sigma_{11}\cos\theta & -\sigma_{22}\sin\theta & 0 \\ \sigma_{11}\sin\theta & \sigma_{22}\cos\theta & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{\sigma}}' = \begin{bmatrix} \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta & (\sigma_{11} - \sigma_{22})\sin\theta\cos\theta & 0 \\ (\sigma_{11} - \sigma_{22})\sin\theta\cos\theta & \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

Use the double-angle identities for sine and cosine to simplify the expressions for the normal stress σ_{11}' and the shear stress σ_{12}' become in the new coordinate system:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\sigma_{11}' = \sigma_n = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2}\cos 2\theta$$
$$\sigma_{12}' = \tau = \frac{\sigma_{11} - \sigma_{22}}{2}\sin 2\theta$$

The expression for the normal stress and shear stress can be shown graphically in shear space:

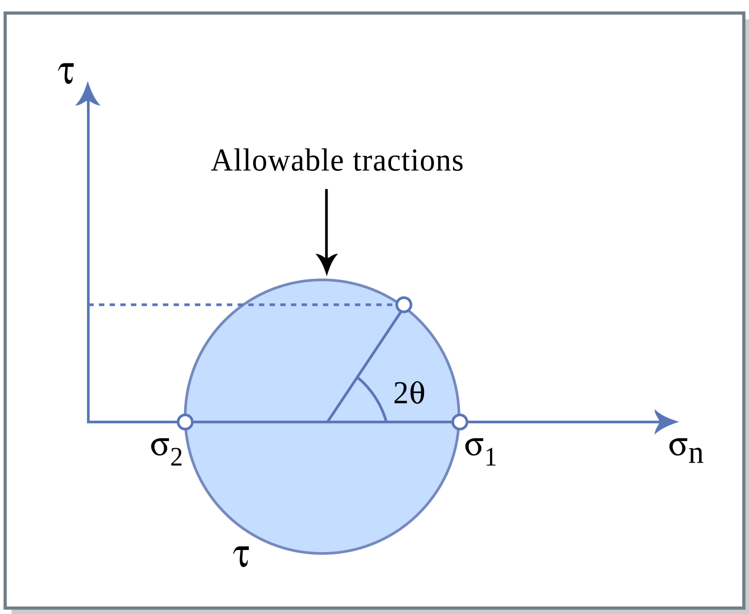

Figure 5.4

Figure by MIT OCW.

This figure is called Mohr's circle.

Interpreting Mohr's circle

Mohr's circle plots in stress space. Admonton's law may also be plotted in stress space as a line with slope f_s. When this line and Mohr's circle intersect, the criterion for failure across a plane is met.

Consider a common experiment in rock mechanics in which scientists apply a uniaxial stress σ_2 to a cylindrical sample confined by a uniform stress σ_1.

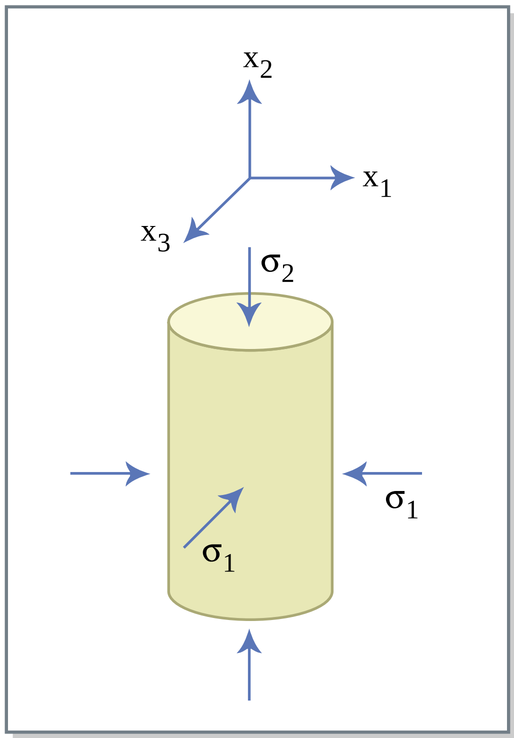

Figure 5.5

Figure by MIT OCW.

Admonton's law and the Mohr's circle that represents the state of stress are plotted below. The Mohr's circle plots on the negative σ_n axis because these notes follow the convention that compressive stresses are negative. σ_1 plots farther to the right because of the convention that σ_1 is greater than σ_2, and σ_2 is greater than σ_3. The line that represents Admonton's law crosses the τ axis at σ_o, the shear strength of the rock.

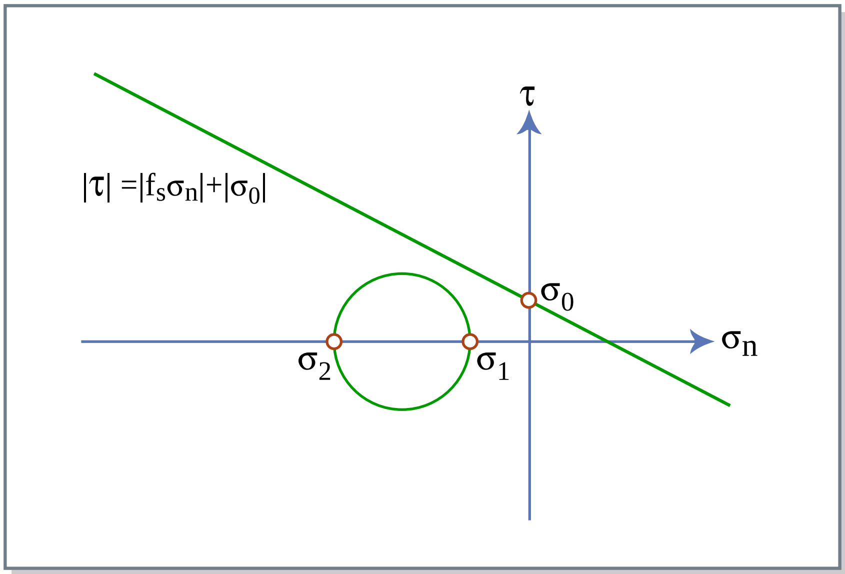

Figure 5.6

Figure by MIT OCW.

As the uniaxial stress σ_2 is increased, the Mohr's circle becomes larger. When the circle intersects the line of Admonton's law, the rock breaks. The angle 2θ at which the circle intersects the line is twice the angle between σ_1 and the normal vector to the failure plane.

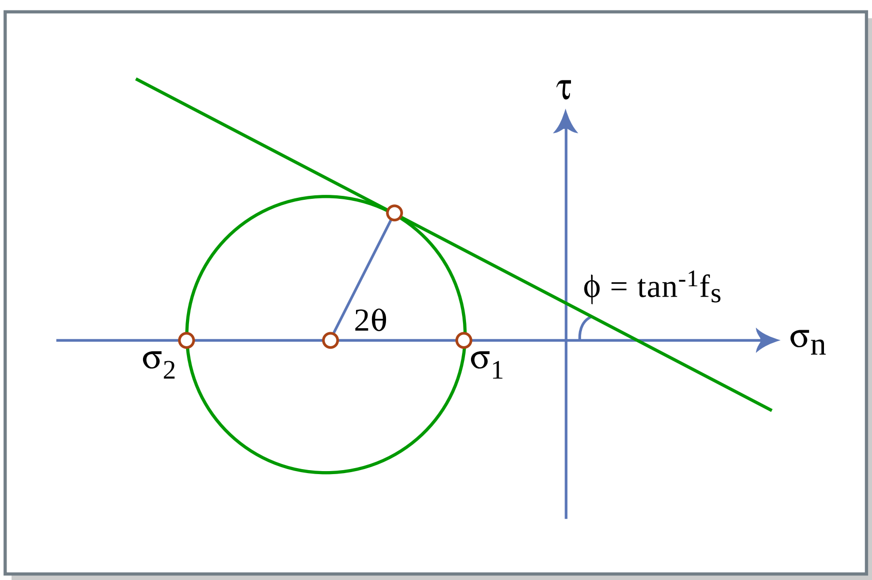

Figure 5.7

Figure by MIT OCW.

The above picture shows that the coefficient of friction determines the slope of Admonton's law, and the slope of Admonton's law determines 2θ. Consequently, predicting the failure plane of a rock only requires knowing the direction of the most compressive stress σ_2, the direction of the least compressive stress σ_1, and the coefficient of friction f_s. The table below lists the angle between σ_1 and the normal vector to the failure plane for different values of f_s.

f_s	θ
0.0	45°
0.6	30°
1.0	23°
$\lim_{f_s \to \infty}$	0°

Since most rocks have a coefficient of friction of about 0.6, the normal vector to the failure plane is typically 30° from the direction of the least compressive stress. Another way of saying this is that the failure plane is 30° from the direction of the most compressive stress.

Faulting

Faults are large-scale failure planes. Since failure planes are normal to the plane containing the least compressive and the most compressive stresses and are typically at 30° from the direction of the most compressive stress, different styles of faulting can be used to infer the directions of σ_1, σ_2, and σ_3.

The following diagram shows the deviatoric stresses associated with thrust faults, normal faults, and strike-slip faults. The pictures assume that x_2 is vertical and that x_1 and x_3 lie in the plane of the surface of the Earth.

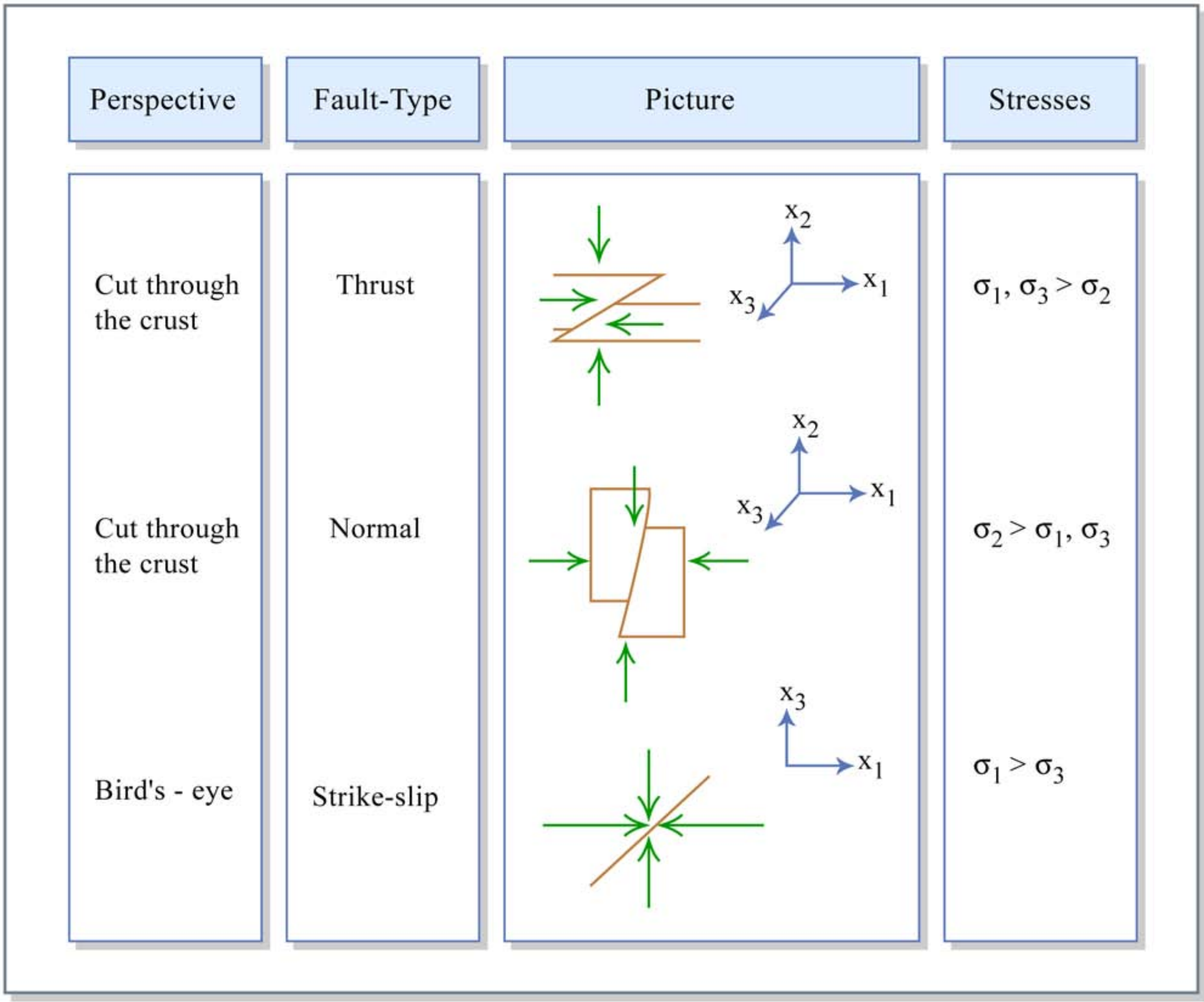

Figure 5.8

Figure by MIT OCW.