12.520 Lecture Notes 22

Fluids (continued)

Material Derivative

Laws of physics – conservation of mass, conservation of energy, etc.

Express in reference frame of material, e.g. rod

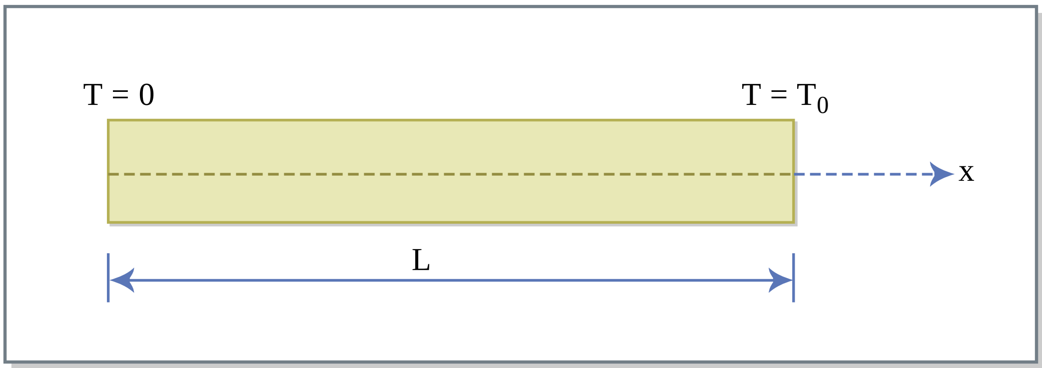

Figure 22.1

Figure by MIT OCW.

Steady state: $T = T_0 x / L; \quad \frac{\partial T}{\partial t} = 0$

Lagrangian frame: $\rho c_p \frac{\partial T}{\partial t} = -k\nabla^2 T + A$

Eulerian frame – material is moving. There would be a $\frac{\partial T}{\partial t}$ for the above rod moving through.

Marching band example.

Need to account for “non physical” change due to motion.

Above example: $\frac{\partial T}{\partial t} = -v\frac{\partial T}{\partial x}$ where $-v\frac{\partial T}{\partial x}$ is advection term.

Material derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \nabla$$

Heat conduction

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \underset{\sim}{v} \nabla T = -k \nabla^2 T + H$$

Conservation of Mass – Continuity Equation

Consider motion in x_2 direction:

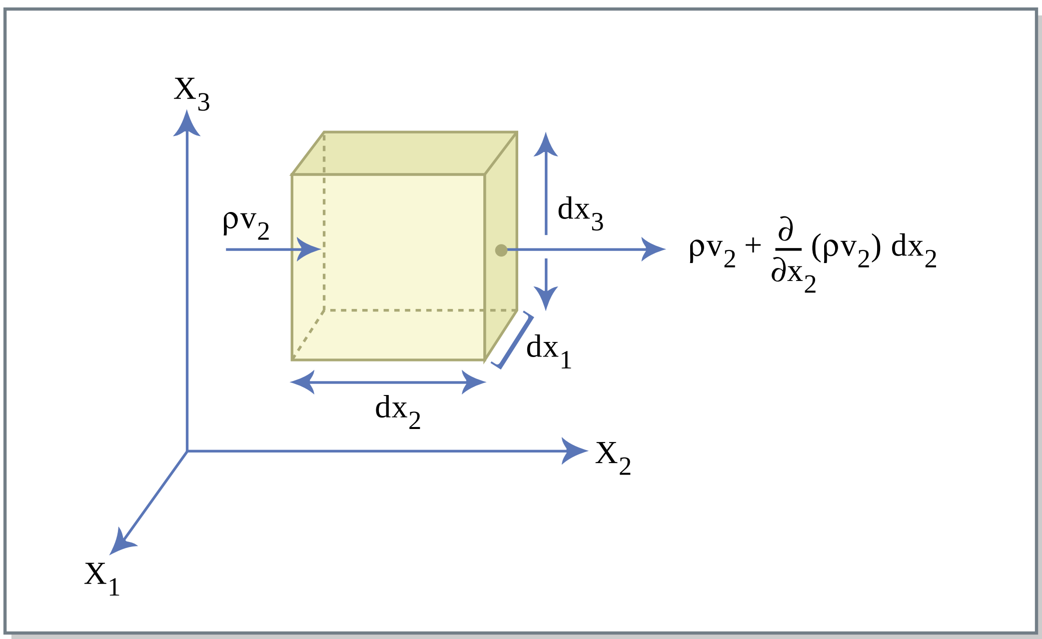

Figure 22.2
Figure by MIT OCW.

Sides: mass in - mass out = $-\frac{\partial}{\partial x_2}(\rho v_2) dx_2 dx_1 dx_3 = -\frac{\partial}{\partial x_2}(\rho v_2) dV$

Front, back: mass in - mass out = $-\frac{\partial}{\partial x_1}(\rho v_1) dV$

Top, bottom: mass in - mass out = $-\frac{\partial}{\partial x_3}(\rho v_3) dV$

For all 3 directions: $-\frac{\partial}{\partial x_1}(\rho v_1)-\frac{\partial}{\partial x_2}(\rho v_2)-\frac{\partial}{\partial x_3}(\rho v_3)=\frac{\partial \rho}{\partial t}$

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho v_i)=0$$

$$\frac{\partial \rho}{\partial t}+v_i\frac{\partial \rho}{\partial x_i}+\rho\frac{\partial v_i}{\partial x_i}=0$$

$$\frac{D\rho}{Dt}+\rho\frac{\partial v_i}{\partial x_i}=0 \quad \text{(Law of conservation of mass)}$$

For an incompressible fluid with constant properties

$$-\nabla p+\mu\nabla^2 \underset{\sim}{v}+\rho \underset{\sim}{x}=\rho\frac{D\underset{\sim}{v}}{Dt}$$

or, with $\nu \equiv \mu/\rho$ (dynamic viscosity)

$$-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 v_i}{\partial x_j \partial x_j}+x_i=\frac{Dv_i}{Dt} \quad \text{(Navier-Stokes equation)}$$

“Plane strain”

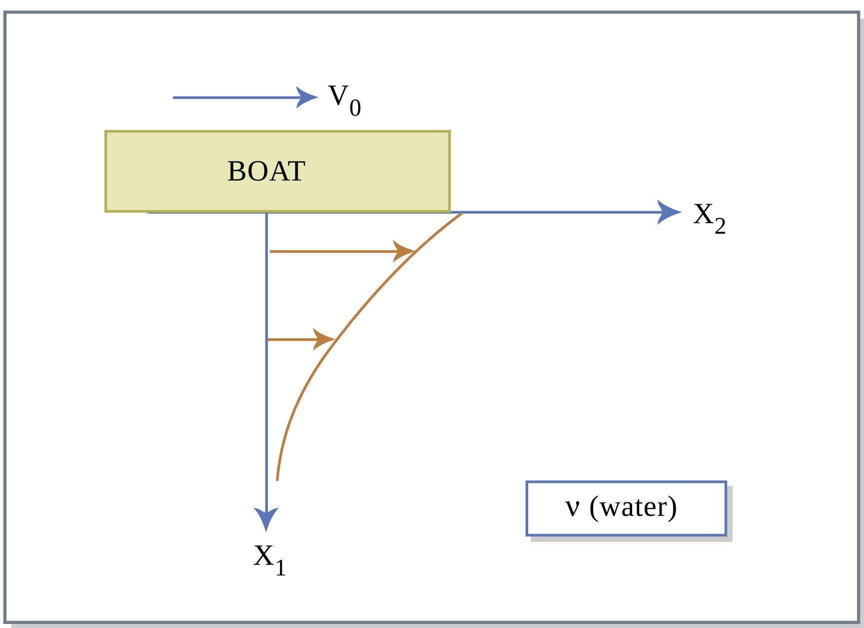

Figure 22.3

Figure by MIT OCW.

$t = 0,\ \underset{\sim}{v} = 0,\ \underset{\sim}{v}(x_1 = 0) = (0, v_0, 0)$

only have $v_2 \neq 0$

∞ in x_2 direction $\Rightarrow\ \dfrac{\partial}{\partial x_2} = 0$

Subtract out hydrostatic

$$\frac{\partial v_2}{\partial t} = \nu \frac{\partial^2 v_2}{\partial x_1^2}$$

The solution becomes $v = v_0(1 - erf\,\dfrac{x_1}{2\sqrt{\nu t}})$

where $erf(y) = \dfrac{2}{\pi}\displaystyle\int_0^y e^{-\xi^2} d\xi$.

Velocity propagates downward a characteristic depth, $x_1 = 2\sqrt{\nu t}$.

Example: canoe 5 meters long,

$v_0 = 5$ m/sec $\Rightarrow t :\ 1\,\text{sec}$

water $\nu :\ 10^{-2}$ cm^2/sec $\Rightarrow$ $x_1 :\ 2\sqrt{10^{-2}} = 2$ mm

A canoe will drag along about 2 mm water.