12.510 Introduction to Seismology
Spring 2008

Introduction to Seismology: Lecture Notes on 17th and 19th March, 2008

The Two Lectures deal with:
1) Interaction of Wave field with Interfaces under Pre-Critical, Critical and Post-Critical Conditions
2) Development of Transmission and Reflection Coefficients

Boundary Conditions:

As discussed in earlier lectures, the equation of motion for a homogenous elastic medium has solutions in which the displacements can be expressed in terms of P,SV,SH Plane Wave potentials as shown below

$$\mathbf{u} = \underbrace{\nabla \phi}_{P} + \underbrace{\nabla \times \nabla \psi_{SV}}_{SV} + \underbrace{\nabla \times \nabla \times \boldsymbol{\psi}_{SH}}_{SH} \ , \nabla \cdot \boldsymbol{\psi}_{SH} = 0$$

(for detailed development see Stein and Wysession)

But, when we have a layered medium, we can take the above solution for homogenous medium for each layer and patch them together at the interfaces to account for the propagation of seismic waves between layers.

Assumption: This works only if the wave front is planar. We implicitly assume that we are far enough from the source to consider the incident wave front as Planar.

There are two types of Boundary Conditions to be considered at every interface:

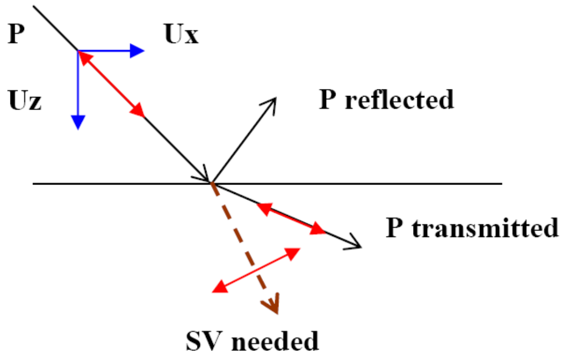
1) Kinematic Boundary Condition (Displacements must be continuous)
2) Dynamic Boundary Condition (Tractions must be Continuous)

Three Principal Interfaces which have to be considered are:

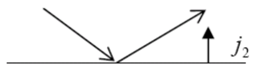
1) Solid-Solid Interface/ Welded Interface (example: Crust-Mantle Interface): .
   All displacements and Tractions have to be continuous
2) Solid-Liquid Interface (example: Ocean Floor, Core-Mantle Boundary): .
   Normal Displacement (i.e Uz) Must to be continuous. .
   Normal Traction ($T_z - \sigma_{xz}, \sigma_{yz}, \sigma_{zz}$) Must be continuous.
   .
   Tangential Displacements need NOT be continuous. .
   Tangential Tractions VANISH. Free slip surface.
3) Free-Surface: .
   All Tractions VANISH. (T_z; $\sigma_{xz}, \sigma_{yz}, \sigma_{zz}$)=0 @ z=0 .
   Displacements are NOT constrained.

Example1: P wave incident on a welded Interface

An incident P wave produces a reflected and a transmitted P wave. But at the interface, the transmitted P wave is not sufficient to preserve the vertical component: a SV wave is needed in order to add sufficient displacement in the vertical component and to satisfy the Boundary Condition. Thus P-SV are coupled.

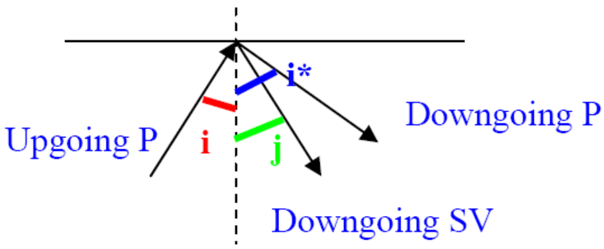


NOTE: In general, P-SV and SH waves are not decoupled. They are only if the medium is isotropic (heterogeneity is permissible) and if the normal at an interface is in the plane of propagation.



Therefore, if the interface dips transverse to the page then the P-SV and SH systems are no longer decoupled (normal no longer lies within the plane of propagation)
Refer: Section 2.5.2 Stein and Wysession for complete development.

Example 2 : P Wave incident on a Free Surface.



From Snell's Law:

$$\frac{\sin i}{\alpha} = \frac{\sin i^*}{\alpha} = \frac{\sin j}{\beta} = p$$

$$\frac{\sin i}{\alpha} = \frac{1}{\alpha_x} = s_x = p$$

Where, α,β are P,S wave velocities in the medium respectively. 'p' is the ray parameter, it is constant for the entire system of rays produced by one incident ray.

Upgoing P wave potential:

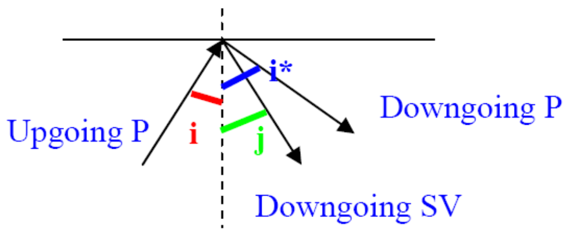
$$\phi'(\underline{x},t) = A\exp\left\{i\omega\left\{\frac{\sin i}{\alpha}.x - \frac{\cos i}{\alpha}.z - t\right\}\right\}$$

Downgoing P wave

$$`\phi(\underline{x},t) = B\exp\left\{i\omega\left\{\frac{\sin i^*}{\alpha}.x + \frac{\cos i^*}{\alpha}.z - t\right\}\right\}$$

Downgoing SV wave

$$`\psi(\underline{x},t) = C\exp\left\{i\omega\left\{\frac{\sin j}{\beta}.x + \frac{\cos j}{\beta}.z - t\right\}\right\}$$



Total Displacement

$$\phi' + `\phi = \phi$$

$$`\psi = \psi \qquad \text{given : } \nabla.\underline{\psi} = 0$$

$$\underline{u} = \nabla\phi + \nabla \times \underline{\psi}$$

Total Stresses:

$$\sigma_{zx} = \sigma_{\phi zx} + \sigma_{\psi zx}$$

$$\sigma_{zz} = \sigma_{\phi zz} + \sigma_{\psi zz}$$

Boundary Conditions:

$\sigma_{zx} = \sigma_{zz} = 0$ free surface: at z=0, $\tau_{zx}=\tau_{zz}=0$ (normal tractions)

Reflection and Transmission Coefficients:

Reflection coefficient: $R \equiv \frac{A_{\text{reflected}}}{A_{\text{incoming}}}$

Transmission coefficient: $T \equiv \frac{A_{\text{transmitted}}}{A_{\text{incoming}}}$

General Approach to solve for Reflection and Transmission Coefficients:

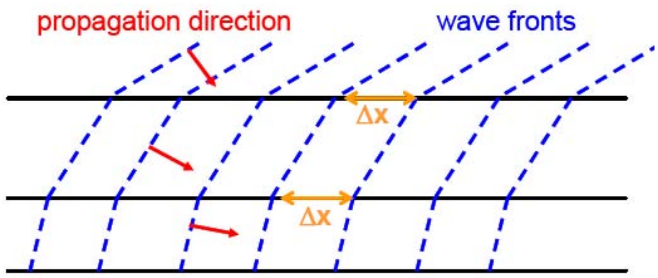
1) Define displacement potentials in the layers on either side of the interface
2) Apply boundary conditions at the interface (Kinematic and Dynamic)
3) Formulate **Zoeppritz** Equations
4) Solve the system of equations
5) Obtain Reflection and Transmission Coefficients

When we have complex layered medium we go for propagator matrix.

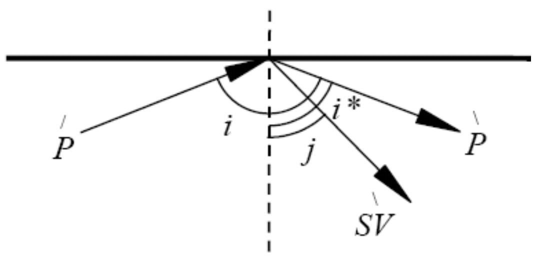
Quick Review of Snell's Law: (refer 2.5.4-2.5.7: Stein and Wysession, for detailed discussion)

Snell's law states that all the waves generated from an incident wave will have the same ray parameter as that of the incident wave.

From a geometric point of view, consider wave propagation across several interfaces as shown below. The wave vector is different in different layers but the horizontal distance travelled along any interface must be the same between the interfaces because the BCs require that wave fronts must be continuous across boundaries. Hence, the apparent horizontal velocity (c_x)is constant everywhere. Since c_x is constant it follows that the horizontal slowness/ ray parameter, p = $1/c_x$, is constant for the entire wave field.



Consider the free surface scenario as shown:



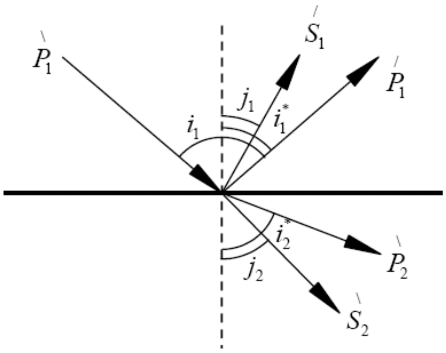
Free Surface Boundary Conditions require that stresses vanish at the free surface.And the stress components contain expressions of the form

$$\omega\left(\frac{\sin i}{\alpha}x - t\right), \omega\left(\frac{\sin i^*}{\alpha}x - t\right), \omega\left(\frac{\sin j}{\beta}x - t\right)$$

As the stress component vanishes, they must all be equal. Hence,

$$\frac{\sin i}{\alpha} = \frac{\sin i^*}{\alpha} = \frac{\sin j}{\beta} \equiv p, \text{ray parameter} \rightarrow i = i^*.$$

Consider the welded interface scenario as shown:



As before, at z =0, the displacement and traction elements contain the following expressions:

$$\left(\frac{\sin i_1}{\alpha_1} - t\right), \left(\frac{\sin i_1^*}{\alpha_1} - t\right), \left(\frac{\sin j_1}{\beta_1} - t\right), \left(\frac{\sin i_2^*}{\alpha_2} - t\right), \left(\frac{\sin j_1}{\beta_2} - t\right),$$

All of which must be equivalent to satisfy the kinematic and dynamic boundary conditions. Hence,

$$\frac{\sin i_1}{\alpha_1} = \frac{\sin i_1^*}{\alpha_1} = \frac{\sin i_2^*}{\alpha_2} = \frac{\sin j_1}{\beta_1} = \frac{\sin j_2}{\beta_2} \equiv p$$

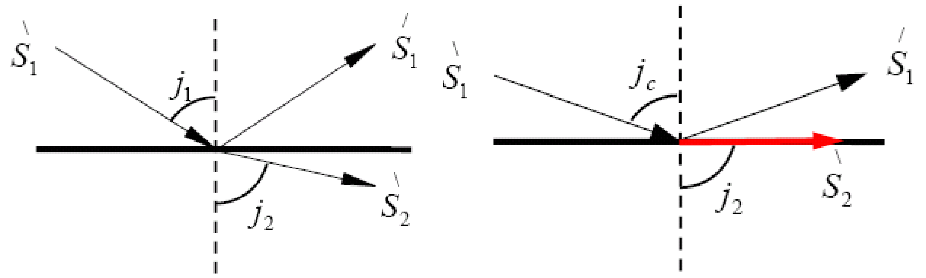
The ray parameter is constant for the entire wave field.

Recall that, $\left(\frac{1}{V}\right)^2 = p^2 + \eta^2$, where V is the velocity of the wave in the medium , 'p' is the horizontal slowness and η is the vertical slowness. Since the velocity of the wave is constant in a medium, and $p = \frac{\sin i}{\alpha}$, which depends on the angle of incidence of the incident wave and is same for all other waves, η can be real or imaginary based on the angle of incidence.

Thus we have three scenarios to consider:

1) Pre-Critical
2) Critical
3) Post Critical

1)Welded Interface SH Case:



We have $p = \frac{\sin j_1}{\beta_1} = \frac{\sin j_2}{\beta_2}$. Given that $\beta_2 > \beta_1$, we have $j_2 > j_1$. So, as we increase j_1 $(at\ j_1 = j_c)$, we have a situation when $j_2 = \frac{\pi}{2}$. This is the Critical Condition.

Pre-Critical: For $j_1 < j_c$, we have $p = \frac{\sin j_1}{\beta_1} < \frac{1}{\beta_2}$. And $\eta_2 = \sqrt{(\frac{1}{\beta_2})^2 - p^2} > 0,\ \eta_2 \epsilon R$

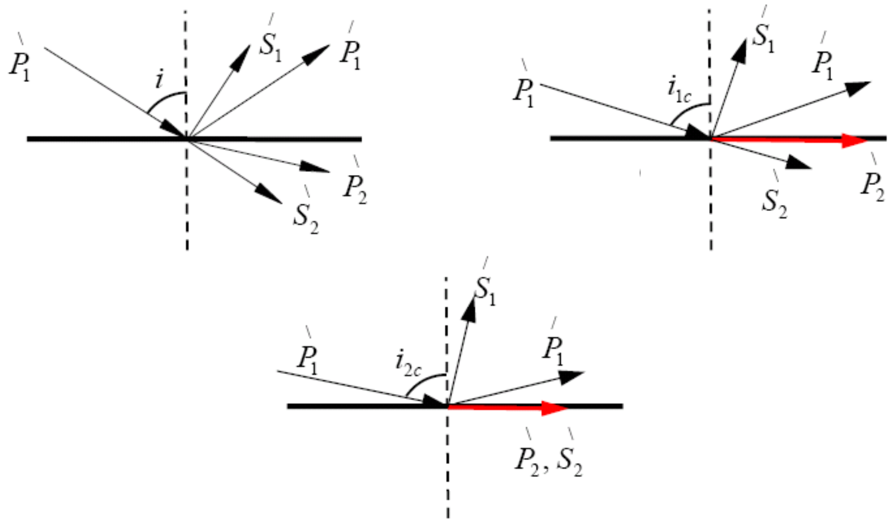
Critical : For $j_1 = j_c$, we have $p = \frac{\sin j_1}{\beta_1} = \frac{1}{\beta_2}$. And $\eta_2 = \sqrt{(\frac{1}{\beta_2})^2 - p^2} = 0$, So we don't have downward propagating wave.

Post-Critical: For $j_1 > j_c$, we have $p = \frac{\sin j_1}{\beta_1} > \frac{1}{\beta_2}$. And

$$\eta_2 = \sqrt{(\frac{1}{\beta_2})^2 - p^2} = i\sqrt{p^2\ (\frac{1}{\beta_2})^2 - \ (\frac{1}{\beta_2})^2} = i\,\eta_2{}^*,\ \eta_2{}^* \epsilon R$$

Thus, η_2 is complex and we have a Evanescent Wave.

2) Welded Surface –Incident P-Wave



Since, P-SV are coupled, we need a SV wave to satisfy the boundary conditions. Let incident P wave make an angle 'i' , transmitted P wave - angle 'j' and transmitted SV wave –angle 'k' with the normal respectively.

We have, $p = \frac{\sin \quad i}{\alpha_1} = \frac{\sin \quad j}{\alpha_2} = \frac{\sin k}{\beta_2}$, also $\alpha_1 < \alpha_2, \beta_1 < \beta_2, (\beta_1,\beta_2) < \ (\alpha_1, \alpha_2)$; Four cases arise in this situation.

Pre-Critical : $i < i_{1c} < i_{2c}$; $p = \frac{\sin i}{\alpha_1} < \frac{1}{\alpha_2} < \frac{1}{\beta_2}$. And so we have,

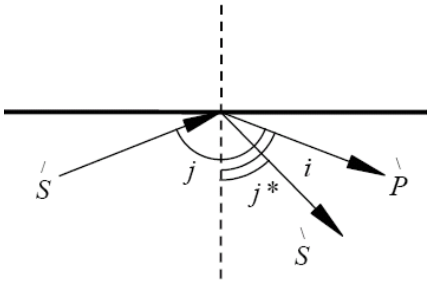
$$\eta_{2\alpha} = \sqrt{(\frac{1}{\alpha_2})^2 - p^2} > 0,\ \eta_{2\alpha} \epsilon R \ \text{ and } \ \eta_{2\beta} = \sqrt{(\frac{1}{\beta_2})^2 - p^2} > 0,\ \eta_{2\beta}\, \epsilon R$$

Critical 1 : $i = i_{1c} < i_{2c}$; $p = \frac{\sin i}{\alpha_1} = \frac{1}{\alpha_2} < \frac{1}{\beta_2}$ and we have $\eta_{2\alpha} = 0 \ and \ \eta_{2\beta} > 0$

Critical 2: $i = i_{2c} > i_{1c}$; $p = \frac{\sin i}{\alpha_1} = \frac{1}{\beta_2} > \frac{1}{\alpha_2}$; we have $\eta_{2\alpha}\ is\ COMPLEX\ and\ \eta_{2\beta} = 0$
Thus, it is post critical for P and critical for SV

Post Critical: : $i > i_{2c} > i_{1c}$; $p = \frac{\sin i}{\alpha_1} > \frac{1}{\beta_2} > \frac{1}{\alpha_2}$; we have $\eta_{2\alpha}, \eta_{2\beta}\ both\ COMPLEX$
Thus, it is critical for both P and SV.

3)Free Surface (P-SV system)



We have $p = \frac{\sin j}{\beta} = \frac{\sin i}{\alpha}$. So, as we increase j_1 ($at\ j_1 = j_c$), we have a situation when $i = \frac{\pi}{2}$.
This is the Critical Condition.

Sub Critical: $j_1 < j_c\,; \eta_\alpha > 0$
Critical for P: $j_1 = j_c\,; \eta_\alpha = 0$ Now it is critical for P
Post Critical : $j_1 > j_c\,; \eta_\alpha\ is\ COMPLEX$

We can have a case when $\frac{1}{\alpha} < \frac{1}{\beta} < p$. This gives rise to Evanescent P,S waves which couple to give LOVE Waves. We shall learn about them in coming lectures.

Reflection and Transmission Coefficients for SH Wave interacting at a Welded Interface:

Define Potentials:

Consider an SH-wave incident at an interface between two layers of different elastic properties. Because SH waves are decoupled from P and SV waves, the transmitted and reflected waves will also be SH waves. Recall that displacement corresponding to SH waves (u_y) satisfies the wave equation. So, SH part doesn't require potentials to find solutions. But we could analyze SH part through potentials too. For simplicity, we deal with displacements directly instead of potentials.