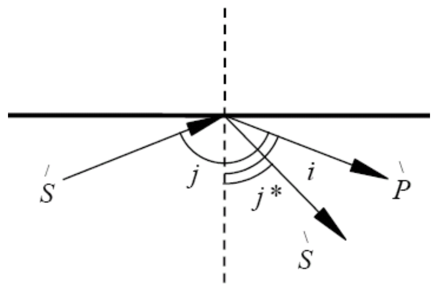
12.510 Introduction to Seismology
Spring 2008

3)Free Surface (P-SV system)



We have $p = \frac{\sin j}{\beta} = \frac{\sin i}{\alpha}$. So, as we increase j_1 $(at\ j_1 = j_c)$, we have a situation when $i = \frac{\pi}{2}$. This is the Critical Condition.

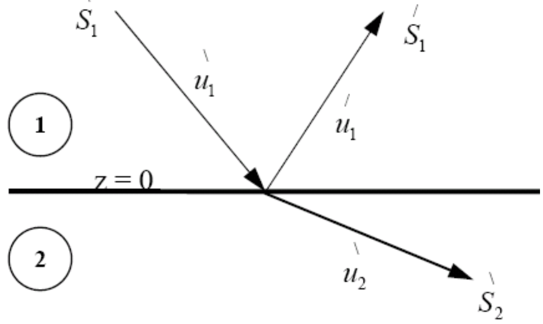
Sub Critical: $j_1 < j_c\ ; \eta_\alpha > 0$
Critical for P: $j_1 = j_c\ ;\ \eta_\alpha = 0$ Now it is critical for P
Post Critical : $j_1 > j_c\ ; \eta_\alpha\ is\ COMPLEX$

We can have a case when $\frac{1}{\alpha} < \frac{1}{\beta} < p$. This gives rise to Evanescent P,S waves which couple to give LOVE Waves. We shall learn about them in coming lectures.

Reflection and Transmission Coefficients for SH Wave interacting at a Welded Interface:

<u>Define Potentials:</u>

Consider an SH-wave incident at an interface between two layers of different elastic properties. Because SH waves are decoupled from P and SV waves, the transmitted and reflected waves will also be SH waves. Recall that displacement corresponding to SH waves (u_y) satisfies the wave equation. So, SH part doesn't require potentials to find solutions. But we could analyze SH part through potentials too. For simplicity, we deal with displacements directly instead of potentials.



The displacement of the down going wave in the Layer1 can be expressed as

$$\grave{u}_1(\mathbf{x},t) = \grave{A}_1\, g(p,\eta_1,\mathbf{x},t)$$

Where for notational convenience

$$\begin{aligned} &g(p,\eta_1,\mathbf{x},t) \equiv \exp[i\omega(px + \eta_1 z - t)] \\ &\eta_1 \equiv \eta_{1,\beta} \\ &\mathbf{x} \equiv (x,z) \end{aligned}$$

And the reflected, up going wave in Layer1 can be expressed as

$$\acute{u}_1(\mathbf{x},t) = \acute{A}_1\, g(p,-\eta_1,\mathbf{x},t)$$

Now, setting the interface to z =0, the total displacement in Layer1 is

$$\acute{u}_1 + \grave{u}_1 = \acute{A}_1\, g(p,\eta_1,\mathbf{x},t) + \grave{A}_1\, g(p,-\eta_1,\mathbf{x},t)$$

The displacement in Layer2 is given by

$$\grave{u}_2 = \grave{A}_2\, g(p,\eta_2,\mathbf{x},t)$$

<u>Applying Boundary Conditions:</u>

The kinematic boundary conditions require that the displacement be continuous across the interface at z = 0. Therefore, setting z = 0 and equating the displacements in Layers 1 and 2, we arrive at the following statement:

$$\acute{u}_1 + \grave{u}_1 = \grave{u}_2 \Leftrightarrow \acute{A}_1 + \grave{A}_1 = \grave{A}_2 \quad (1)$$

The dynamic boundary conditions require that tractions are continuous. We have,

$$\sigma_{ij} = \lambda\delta_{ij}\Delta + 2\mu\varepsilon_{ij} = \lambda\delta_{ij}\Delta + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

And recalling, for SH waves, $\mathbf{u} = (0, \mathrm{u_y}, 0)$, we have:

$$\sigma_{xz} = 0,\ \ \sigma_{yz} = \mu\frac{\partial u_y}{\partial z},\ \ \sigma_{zz} = 0$$

$$\grave{\sigma}_{yz,1} = \mu_1 \frac{\partial \grave{u}_{y,1}}{\partial z} = i\omega\mu_1\eta_1 \grave{A}_1\, g(p,\eta_1,\mathbf{x},t)$$

$$\acute{\sigma}_{yz,1} = \mu_1 \frac{\partial \acute{u}_{y,1}}{\partial z} = -i\omega\mu_1\eta_1 \acute{A}_1\, g(p,-\eta_1,\mathbf{x},t)$$

$$\grave{\sigma}_{yz,2} = \mu_1 \frac{\partial \grave{u}_{y,2}}{\partial z} = i\omega\mu_2\eta_2 \grave{A}_2\, g(p,\eta_2,\mathbf{x},t)$$

hence, @z=0,

$$\sigma_1 = \acute{\sigma}_1 + \grave{\sigma}_1 = i\omega\mu_1\eta_1\left(\grave{A}_1 - \acute{A}_1\right)\exp[i\omega(px-t))]$$

$$\sigma_2 = \grave{\sigma}_2 = i\omega\mu_2\eta_2 \grave{A}_2 \exp[i\omega(px-t))]$$

The preceding facts imply that

$$\mu_1\eta_1\left(\grave{A}_1 - \acute{A}_1\right) = \mu_2\eta_2 \grave{A}_2 \ \ldots(2)$$

Formulating Zoeppritz Equations:

The reflection and transmission coefficients are expressed as

$$R_{ss} = \grave{S}\acute{S} = \frac{\acute{A}_1}{\grave{A}_1}$$

$$T_{ss} = \grave{S}\grave{S} = \frac{\grave{A}_2}{\grave{A}_1}$$

If we arrange Eqn 2

$$\acute{A}_1 - \grave{A}_1 = -\grave{A}_2 \frac{\mu_2\eta_2}{\mu_1\eta_1}$$

Combining eqns 1 and 2 give the **Zoeppritz** equation

$$\grave{A}_1\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1 & 1\\ 1 & \dfrac{\mu_2\eta_2}{\mu_1\eta_1}\end{pmatrix}\begin{pmatrix}\acute{A}_1\\ \grave{A}_2\end{pmatrix}$$

Where the left hand side is the incoming wave and the RHS are the resulting waves.

<u>Solving the Equations for Reflection and Transmission Coefficients:</u>

Again, if we arrange the **Zoeppritz** equation:

$$\left(\acute{A}_1 - \grave{A}_1\right)\mu_1\eta_1 = -\left(\grave{A}_1 + \acute{A}_1\right)\mu_2\eta_2$$

$$\acute{A}_1\left(\mu_1\eta_1 + \mu_2\eta_2\right) = \grave{A}_1\left(\mu_1\eta_1 - \mu_2\mu_2\right)$$

Then the reflection coefficient is

$$R \equiv \frac{\acute{A}_1}{\grave{A}_1} = \frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}$$

And to calculate the transmission coefficient:

$$\grave{A}_2 = \grave{A}_1 + \acute{A}_1 = \grave{A}\left(1 + \frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}\right) = \grave{A}_1\left(\frac{2\mu_1\eta_1}{\mu_1\eta_1 + \mu_2\eta_2}\right)$$

$$T = \frac{\grave{A}_2}{\grave{A}_1} = \frac{2\mu_1\eta_1}{\mu_1\eta_1 + \mu_2\eta_2}$$

The dependence on the incidence angle is illustrated by

$$\eta_1 = \frac{\cos i_1}{\beta_1}, \eta_2 = \frac{\cos i_2}{\beta_2}$$

And

$$\beta^2 = \frac{\mu}{\rho}$$

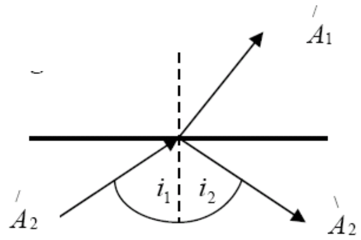
Using these variables to rewrite the reflection and transmission coefficients:

$$R = \frac{\rho_1\beta_1 \cos i_1 - \rho_2\beta_2 \cos i_2}{\rho_1\beta_1 \cos i_1 + \rho_2\beta_2 \cos i_2} = \frac{\rho_1\beta_1 - \rho_2\beta_2}{\rho_1\beta_1 + \rho_2\beta_2}$$

$$T = \frac{2\rho_1\beta_1 \cos i_1}{\rho_1\beta_1 \cos i_1 + \rho_2\beta_2 \cos i_2} = \frac{2\rho_1\beta_1}{\rho_1\beta_1 + \rho_2\beta_2}$$

The reflection and transmission coefficients are equal to the RHS of the equations above for the specific case of normal incidence, when $i_1 = i_2 = 0$.

For a welded surface we can also have $\acute{A}_2 \to \acute{A}_1 and\ \grave{A}_2$



Thus we can define a scatter matrix S

$$\underline{\underline{S}} = \begin{pmatrix} \acute{S}\acute{S} & \grave{S}\acute{S} \\ \acute{S}\grave{S} & \grave{S}\grave{S} \end{pmatrix}$$

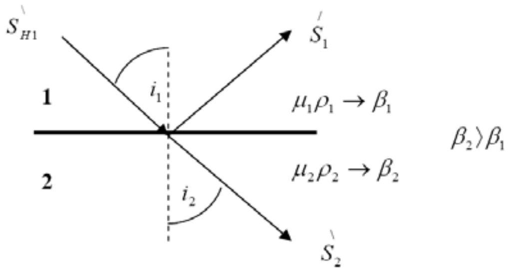
Behaviour of Reflection and Transmission Coefficients with angle of incidence:

We have,

$$R \equiv \frac{\acute{A}_1}{\grave{A}_1} = \frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}$$

$$T = \frac{\grave{A}_2}{\grave{A}_1} = \frac{2\mu_1\eta_1}{\mu_1\eta_1 + \mu_2\eta_2}$$

As we have discussed before, since η_1 and η_2 are functions of ray parameter, R and T depend upon the incident angle.



Say i_c is the incident angle for which $i_2 = \frac{\pi}{2}$. So, i_c is the critical angle

Pre-Critical: $i_1 < i_c$. Now, we can express η_1 *and* η_2 as below:

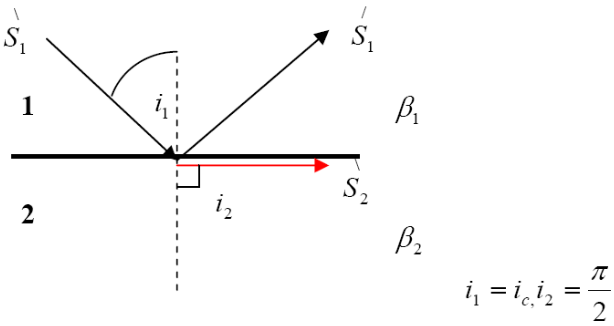
$$\eta_1 = \frac{\cos i_1}{\beta_1}, \eta_2 = \frac{\cos i_2}{\beta_2}$$

Substituting the above in the relation for R and T, we have:

$$R = \frac{\rho_1\beta_1 \cos i_1 - \rho_2\beta_2 \cos i_2}{\rho_1\beta_1 \cos i_1 + \rho_2\beta_2 \cos i_2} \qquad T = \frac{2\rho_1\beta_1 \cos i_1}{\rho_1\beta_1 \cos i_1 + \rho_2\beta_2 \cos i_2}$$

Let's assume that $\rho_2\beta_2 > \rho_1\beta_1$. So, when $i_1 = i_2 = 0$, we have R<0 and 0<T<1. But, as we keep increasing the angle of incidence, we have a situation when $\rho_2\beta_2 \cos i_2 = \rho_1\beta_1 \cos i_1$. At this point we have R = 0 and T=1. This is called the **angle of intromission.** Beyond this angle, $\rho_2\beta_2 \cos i_2 < \rho_1\beta_1 \cos i_1$ and we have R>0 and T >1 until we reach the critical angle.

Critical Angle: If we increase the angle such that $i_1 = i_c$ and we have $\eta_2 = 0$. Thus, R = 1 and T=2.



$$p = \frac{\sin i_1}{\beta_1} = \frac{\sin i_c}{\beta_1} = \frac{\sin i_2}{\beta_2} = \frac{1}{\beta_2} \Rightarrow \sin i_c = \frac{\beta_1}{\beta_2}$$

$$\eta_\beta = \sqrt{\frac{1}{(\beta_2)^2} - \frac{1}{(\beta_2)^2}} = 0 \text{ therefore there is no vertical propagation, } k_z = 0$$

$$R_{\grave{S}\acute{S}} = \grave{S}\acute{S} = \frac{\rho_1\beta_1\cos i_1 - \rho_2\beta_2\cos i_2}{\rho_1\beta_1\cos i_1 + \rho_2\beta_2\cos i_2} = 1 \quad \text{, a full reflection}$$

$$T_{\grave{S}\grave{S}} = \grave{S}\grave{S} = \frac{2\rho_1\beta_1\cos i_1}{\rho_1\beta_1\cos i_1 + \rho_2\beta_2\cos i_2} = 2 \quad \text{(also known as the Head Wave)}$$

Post-Critical Angle: If we increase the angle beyond i_c , i_2 can't further change and remains at $\frac{\pi}{2}$. Now we have:

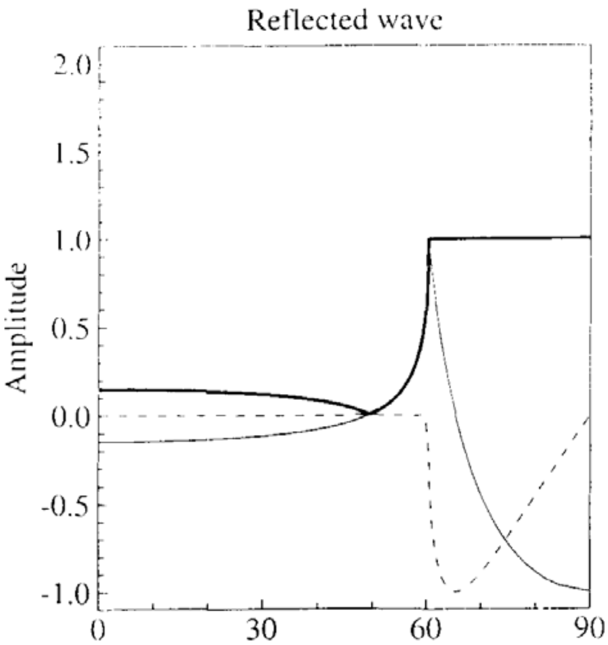
$$\frac{\sin i_1}{\beta_1} \rangle \frac{\sin i_c}{\beta_1} = p = \frac{1}{\beta_2}$$

For the incoming wave, $p > {}^1/_{\beta_2}$. There fore,

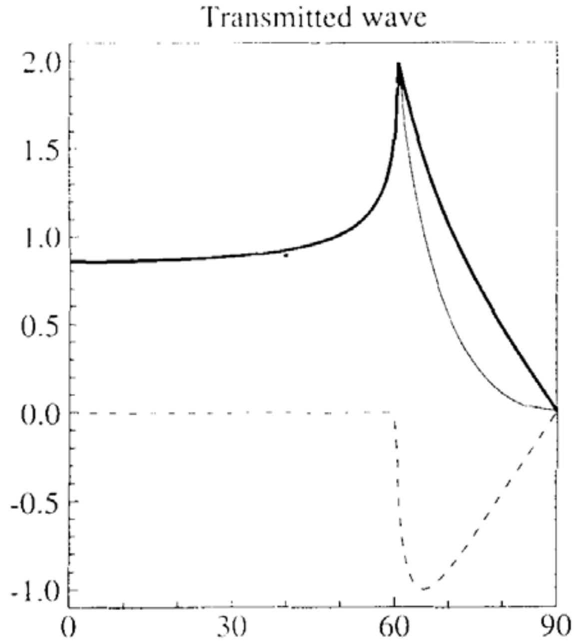
$$\eta_2 = \sqrt{\frac{1}{(\beta_2)^2} - p^2} = i\sqrt{p^2 - \frac{1}{(\beta_2)^2}} = i\hat{\eta}_2$$

η_2 is now a Complex Number.

Since, η_2 is complex, we have both R and T to be Complex. R and T are plotted as a function of the incidence angle (see figure below). The real part is in thin solid line, imaginary part in dashed line and absolute value in bold line.



Incidence angle in Degrees



Incidence angle in Degrees

<u>Consider the transmitted wave:</u>

For a SH wave, the displacement vector is $\mathbf{u} = (0, \mathrm{u_y}, 0)$

$$U_{y,2} = \exp\{i\omega(px + \eta_2 z - t)\}$$

In a post critical situation when $\eta_2 = i\,\hat{\eta}_2$,

$$U_{y,2} = \exp\{iw(px + i\hat{\eta}_2 z - t)\}$$
$$U_{y,2} = \exp\{-\omega\hat{\eta}_2 z\}\exp\{i\omega(px - t)\}$$

Where the first term on the RHS describes an exponential decay in the z-direction (there is no propagation in the z-direction). The frequency, ω, controls the rate of the decay. The second term on the RHS describes a harmonic function (**x**,t), therefore the wave propagation is in the x-direction.

$$U_{y,2} = A(z)\exp\{i\omega(px - t)\}$$

The property of deceasing wave amplitude with depth based on the frequency of the wave is known as the Evanescence.

$$\omega\eta = k_z = \frac{2\Pi}{\lambda}$$

$$\eta = \frac{2\Pi}{\lambda\omega}$$

A wave with a short wavelength (λ), high frequency (ω), will decay more quickly. At infinite frequency the decay is instantaneous and the wave becomes a ray.

If the properties of a medium change with depth, for example there is a body which allows a wave to pass through it more quickly at depth only the low ω , long λ, waves will sample it as the high waves will have been stripped out. A wave with frequency dependence is a **Dispersive wave**. Often a wave can be dispersive and evanescent.

Consider the Reflected Wave:

The reflection coefficient is $R = \frac{\mu_1\eta_1 - i\,\mu_2\hat{\eta}_2}{\mu_1\eta_1 + i\,\mu_2\hat{\eta}_2} = \frac{e^{-i\theta}}{e^{+i\theta}} = e^{-i(2\theta)}$

Where, $\theta = tan^{-1}\left(\frac{\mu_2\hat{\eta}_2}{\mu_1\eta_1}\right)$

Now, if we write the displacements of the reflected wave,

$$\begin{aligned} \acute{U}_{y,1} &= R.A_i e^{i(\boldsymbol{k}.\boldsymbol{x}-\omega.t)} \\ &= A_i\, e^{-i(2\theta)} e^{i(\boldsymbol{k}.\boldsymbol{x}-\omega.t)} \\ &= A_i e^{i(\boldsymbol{k}.\boldsymbol{x}-\omega(t+\frac{2\theta}{\omega}))} \end{aligned}$$

Where, A_i is the amplitude of the incident wave.

Thus, we can observe that the reflected wave has a phase shift of $\frac{2\theta}{\omega}$. For a given incident angle, as we observe, different frequencies have different phase shifts. So, a single incident spike would be recorded as a spread waveform at the receiver as different frequencies are shifted apart in phase.

References and Sources of figures:

Introduction to Seismology: Stein and Wysesion

Class Notes on 2/23/05: Compiled by Sophie Michelet
Class Notes on 2/28/05
Class Notes on 3/2/05

Introduction to Seismology: Lecture Notes on 17th and 19th March, 2008

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