Chapter 5

Large-scale motions on a rotating Earth

5.1 The equations of motion on a rotating plane

In an inertial frame of reference, the equation of motion (momentum) is

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p - \rho \nabla \Phi + \mathbf{F} \tag{5.1}$$

where $\mathbf{u}$ is the vector velocity, $\Phi = gz$ the gravitational potential, and $\mathbf{F}$ represents any applied external body force or frictional forces acting per unit volume. For a tidal problem, $\mathbf{F}$ would represent the gravitational tidal force; in the ionized upper atmosphere, it could include forces involved in moving ions across the Earth's magnetic field lines. In almost all cases, however, such effects are negligible, and the only "force" acting is friction. Even this is negligible, except very close to the Earth's surface (for the atmosphere), or in the surface and bottom (benthic) boundary layers of the ocean.

Expressed relative to a frame rotating with the planetary rotation rate $\mathbf{\Omega}$, equation (5.1) is

$$\rho \left(\frac{d\mathbf{u}}{dt} + 2\mathbf{\Omega} \times \mathbf{u} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \right) = -\nabla p - \rho \nabla \Phi + \mathbf{F} \ ,$$

the 2nd and 3rd terms on the LHS representing the Coriolis and centrifugal terms, respectively, and $\mathbf{r}$ is the position vector measured from the planetary center. It is conventional to simplify things a little by absorbing the

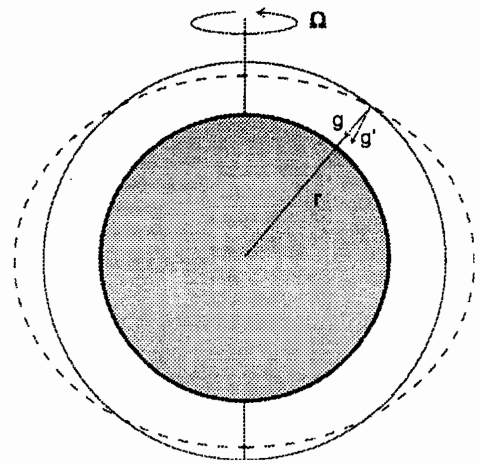

Figure 5.1: Geometry of a spherical surface (solid) and the geoid (dashed) through a point at location r.

centrifugal term into the gravitational potential. One can do this easily, since

$$\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) = \nabla(\frac{1}{2}\Omega^2 r^2) ,$$

where $\Omega = |\mathbf{\Omega}|$ and $r = |\mathbf{r}|$; hence one can absorb this term into the definition of Φ (writing $\widetilde{\Phi} = \Phi - \frac{1}{2}\Omega^2 r^2$), leaving

$$\rho\left(\frac{d\mathbf{u}}{dt} + 2\mathbf{\Omega}\times\mathbf{u}\right) = -\nabla p - \rho\nabla\widetilde{\Phi} + \mathbf{F} . \tag{5.2}$$

We now have to regard gravity ($\nabla\widetilde{\Phi}$) not as g, pointing downward relative to the spherical surface through $\mathbf{r}$, but as g', pointing downward relative to the **geoid** through $\mathbf{r}$ (see Fig. 5.1). So in order for gravity to remain vertical we must, in principle, use slightly non-spherical coordinates; in practice, the geoid is so close to being spherical that we can ignore this complexity without introducing significant error. Thus, we ignore the "twiddle" on Φ in (5.2).

5.2 Rapid rotation

The material time derivative on the LHS of (5.2) can be written

$$\frac{d\mathbf{u}}{dt} + 2\mathbf{\Omega}\times\mathbf{u} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + 2\mathbf{\Omega}\times\mathbf{u} .$$

If we assume that a typical magnitude for velocity is U, that the distance on which velocity varies is typically L, and the time on which it changes is typically T, the 3 terms on the RHS have typical magnitudes

$$\frac{U}{T}\ ;\ \frac{U^2}{L}\ ;\ 2\Omega U\ .$$

For motions that are nearly **steady**, in the sense that $2\Omega T \gg 1$, and are **slow** in the sense that $Ro \equiv U/(2\Omega L) \ll 1$, the third term (the Coriolis term) is dominant. (The dimensionless number Ro is known as the **Rossby number** of the flow.) Now, the rotation rate $\Omega = 2\pi/(1\text{day}) = 7.2722\ 10^{-5}\text{s}^{-1}$. The first condition requires $T \gg 0.08$day, an excellent approximation for large scale motions in the atmosphere ($T \geq 1$day) and even better for large-scale motions in the ocean. As for the second condition, for a synoptic system in the atmosphere, $U \simeq 30\text{ms}^{-1}$, while $L \simeq 1000$km, so $Ro \simeq 0.2$; for an oceanic eddy, $U \simeq 0.1\text{ms}^{-1}$, $L \simeq 100$km, so $Ro \simeq 0.01$ (and for larger scale motions, it is smaller than this). So the assumption $Ro \ll 1$ is quite good for large scale motions in the atmosphere and excellent for the ocean.

If we assume the Coriolis term to dominate the LHS, therefore, and further assume that the motions are **inviscid** (so that $\mathbf{F} = 0$, an excellent approximation outside boundary layers), eq. (5.2) becomes

$$2\mathbf{\Omega}\times\rho\mathbf{u} = -\mathbf{\nabla}p - \rho\mathbf{\nabla}\Phi\ .$$

Taking the curl (since $\nabla \times \nabla a = 0$, for any a), we get

$$\nabla \times (2\mathbf{\Omega}\times\rho\mathbf{u}) = -\nabla \times (\rho\mathbf{\nabla}\Phi)\ .$$

Using the vector identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A}) - (\mathbf{A}\cdot\nabla)\,\mathbf{B} + (\mathbf{B} \cdot \nabla)\,\mathbf{A}\ ,$$

and noting that $\mathbf{\Omega}$ is constant, we get

$$\nabla \times (2\mathbf{\Omega}\times\rho\mathbf{u}) = 2\mathbf{\Omega}\,(\nabla \cdot [\rho\mathbf{u}]) - 2\,(\mathbf{\Omega} \cdot \nabla)\,\rho\mathbf{u}\ .$$

But continuity of mass gives

$$\frac{\partial\rho}{\partial t} + \nabla \cdot \rho\mathbf{u} = 0\ .$$

So, for steady flow, $\nabla \cdot [\rho \mathbf{u}] = 0$. Thus, using the further vector identity

$$\nabla \times (c\mathbf{A}) = \nabla c \times \mathbf{A} + c\nabla \times \mathbf{A} ,$$

we obtain

$$2\,(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\,\rho\mathbf{u} = \nabla \rho \times \nabla \Phi .$$

If the flow is hydrostatic, then $\partial p/\partial z = -g\rho$, or, since $d\Phi = g\,dz$, $\partial p/\partial \Phi = -\rho$, whence $\nabla \Phi = -\rho^{-1}\nabla p$, and then

$$2\,(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\,\rho\mathbf{u} = -\frac{1}{\rho}\nabla \rho \times \nabla p .$$

Now, if the flow is also **barotropic**, by which we mean that density is a function of pressure only (i.e., no density variations along the almost-horizontal pressure surfaces), $\rho = \rho(p)$, whence $\nabla \rho = \frac{d\rho}{dp}\nabla p$, and so $\nabla \rho \times \nabla p = \frac{d\rho}{dp}\,(\nabla p \times \nabla p) = 0$. Thus we arrive at **the Taylor-Proudmann theorem**:

$$(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\,\rho\mathbf{u} = 0\ :$$

For slow, steady, inviscid, barotropic motions in a rotating system, the momentum density vector $(\rho\mathbf{u})$ *is constant along the direction parallel to the axis of rotation.*

Now, neither the atmosphere nor ocean are truly barotropic (they would be much less interesting if they were) but, nevertheless, many aspects of their dynamics can be captured in models that are barotropic (or nearly so), which is where we start.

5.3 Two-dimensional rotating flow

5.3.1 The barotropic equations of motion

We now investigate the properties of two-dimensional flow. If the atmosphere or ocean is assumed to be barotropic, the flow is independent of the direction along the rotation axis; given that they are also **thin**, in the sense that their depths are very much less than the Earth's radius, it is a very good approximation to assume that this implies that the flow is independent of z, the coordinate vertical to the Earth's surface (see Fig. 5.2). If we adopt

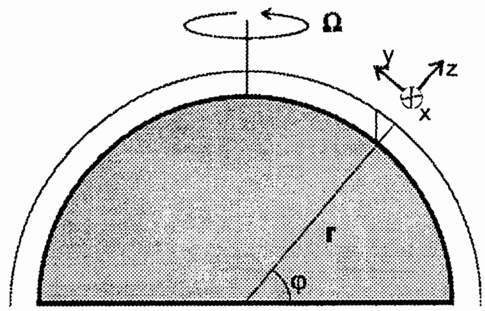

Figure 5.2: Coordinates for a shallow atmosphere or ocean.

local Cartesian coordinates[1] (x, y, z) as shown on the Figure, the flow is in the $(x - y)$ plane, and $w = 0$. The x and y components of (5.2), are then

$$\begin{aligned} \frac{du}{dt} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + G_x \\ \frac{dv}{dt} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + G_y \end{aligned} \tag{5.3}$$

where $f = 2\Omega \sin\varphi$,with φ being latitude, and $\mathbf{G} = (G_x, G_y) = \mathbf{F}/\rho$ is the applied (frictional) force expressed in units of acceleration. Note that the coefficient f appearing in the Coriolis term is twice the vertical (z-) component of the rotation rate. This coefficient is known as the **Coriolis parameter**. Note that f is a function of latitude, the importance of which we shall see later.

Eqs. (5.3) give us 2 equations in the 3 unknowns u, v, and w (ρ is assumed to be known as a function of p). We close the system with the equation of continuity $\nabla \cdot \mathbf{u} = 0$, or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\ . \tag{5.4}$$

5.3.2 Vorticity and the barotropic vorticity equation

One of the difficulties of working with momentum (or velocity) of a parcel in fluid mechanics stems from the pressure forces to which the parcel is

[1] We are assuming here that the region of interest is a small part of the whole globe, otherwise it is necessary to use spherical coordinates (of course).

subjected, which are continuously changing the parcel's momentum in complicated ways (since pressure is not fixed, but itself evolves with the flow). However, while pressure gradients can change a parcel's *momentum*, they cannot change its *spin*, because, as we have seen, for barotropic flow for which[2] $\rho = \rho(p)$,

$$\nabla \times \left(\frac{1}{\rho}\nabla p\right) = 0 .$$

So, if we take the curl of the momentum equations, the pressure gradient term disappears. If we do this for eqs. (5.3), by taking $\partial/\partial x$ of the second minus $\partial/\partial y$ of the first, we get

$$\frac{\partial}{\partial x}\left(\frac{dv}{dt}\right) - \frac{\partial}{\partial y}\left(\frac{du}{dt}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{df}{dy} = Z , \tag{5.5}$$

where $Z = \partial G_y/\partial x - \partial G_x/\partial y$ is the (vertical component of the) curl of the frictional force per unit mass, and note that $f = f(y)$, since it is a function of latitude only. Now, from (5.4), the third term vanishes; moreover, a little mathematical juggling [expand the total derivatives, and use (5.4)] shows that

$$\frac{\partial}{\partial x}\left(\frac{dv}{dt}\right) - \frac{\partial}{\partial y}\left(\frac{du}{dt}\right) = \frac{d}{dt}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) .$$

The term inside the bracket on the RHS is the vertical component of the **vorticity**, defined by

$$\boldsymbol{\xi} = \nabla \times \mathbf{u} ; \tag{5.6}$$

its vertical component is

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} . \tag{5.7}$$

Since the flow in this barotropic problem lies within horizontal planes, only the vertical component is nontrivial[3].

The vorticity is a *local* measure of the spin of the fluid motion. For example if the fluid (relative to the rotating frame, remember) is in solid body rotation about the origin with angular frequency ω, then (see Fig. 5.3)

$$u = -\omega y; v = \omega x ;$$

[2]Note that this includes an fluid of constant density.

[3]In large-scale meterology and oceanography, the general term "vorticity" is often used to mean the vertical component, unless specified otherwise.

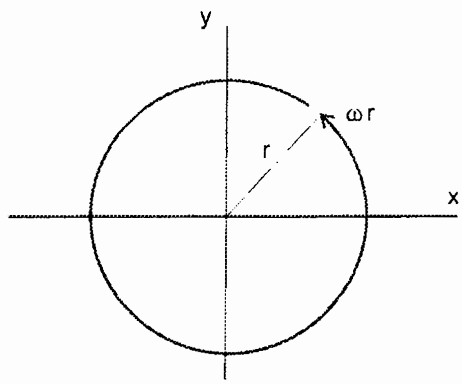

Figure 5.3: Rotation about the origin; the velocity at position $r = (x, y)$ is $U = \omega r$.

so the vorticity is $\zeta = 2\omega$—twice the rotation rate (anticlockwise being positive).

To return to (5.5), then, we have

$$\frac{d\zeta}{dt} = -v\frac{df}{dy} + Z \; . \tag{5.8}$$

This equation states that the time derivative *following the motion* of the vorticity is (in this barotropic case) given by two terms. The second represents the creation or destruction of vorticity by viscous torques (curl of the frictional force per unit mass), while the first represents advection of f, the Coriolis parameter. But we have already seen that $f = 2\Omega \sin \varphi$ is twice the vertical component of the planetary rotation rate; so looking down on the planet at latitude φ, an observer in an inertial frame would say the rotation rate of the fluid is not ω, but $\Omega \sin \varphi + \omega$, and hence that the **absolute vorticity** of the flow—that observed from a nonrotating frame—is not ζ but

$$\zeta_a = f + \zeta \; . \tag{5.9}$$

Now, since f is a function of y only, its material derivative is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = v\frac{\partial f}{\partial y}$$

and so the **barotropic vorticity equation** can be written

$$\frac{d\zeta_a}{dt} = Z \; . \tag{5.10}$$

Therefore:

In inviscid barotropic flow, the absolute vorticity is conserved following the motion.

For most purposes away from boundary layers, the inviscid limit is a relevant one, so this theorem is profoundly useful for barotropic flows. (As we shall see, it needs modification for non-barotropic flows.) Put very simply, it says that if a fluid parcel is at position $\mathbf{x}_o$ and has absolute vorticity ζ_{ao} at time t_0, and moves without viscous influence to position $\mathbf{x}_1$ at time t_1, we know its absolute vorticity is still ζ_{a0}—and we know this without needing to know anything about the path the parcel took in the intervening period. (So absolute vorticity is a **tracer**, and behaves just like, say, a dye marker.) Contrast this with velocity: to know how the velocity changed between t_0 and t_1, we would need to know its path, and the history of the pressure gradient along this path.

Now, one might object that absolute vorticity is not as interesting as velocity—that we may know what it is, but that that knowledge is not useful in telling us about what we want to know. However, if we know the distribution of ζ_a at any time, we know the distribution of ζ (since we know f) and, from that, we can determine the flow. To see this, first note from the continuity equation (5.4) that we can satisfy this by defining velocity in terms of a **stream function** ψ, such that $\mathbf{u} = -\widehat{\mathbf{z}} \times \nabla\psi$, or

$$u = -\frac{\partial\psi}{\partial y} \; ; \; v = \frac{\partial\psi}{\partial x} \; , \tag{5.11}$$

which guarantees that $\nabla \cdot \mathbf{u} = 0$. Since $\mathbf{u}$ is normal to $\nabla\psi$, it is directed along contours of ψ, as shown in Fig. 5.4. Moreover, ψ is a measure of the **flux** of fluid since the net amount of fluid passing per unit between the two streamlines A and B, on which the streamfunction is (say) ψ and $\psi + \delta\psi$ is $|\mathbf{u}|\,\delta l$, where δl is the distance AB between the streamlines. But, from (5.11), $|\mathbf{u}| = |\delta\psi|\,/\delta l$, so the flux (which in this 2-dimensional case has units of area per unit time) between the streamlines is just $|\delta\psi|$. Note that this flux is constant along the streamlines, so the velocity is large where the streamlines are close together, and weak where they are far apart—as is obvious from (5.11).

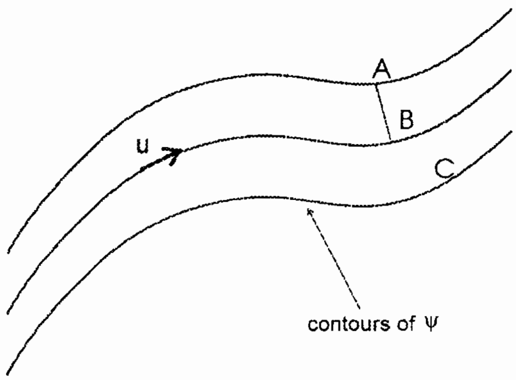

Figure 5.4: Flow is along streamlines (lines of constant ψ).

Now, in terms of streamfunction, it is obvious from (5.7) and (5.11) that the vorticity can be written

$$\zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \equiv \nabla_2^2 \psi \tag{5.12}$$

where ∇_2^2 is the two-dimensional (horizontal) Laplacian operator. So, if we know the vorticity distribution at any time [and note that (5.12) is a *diagnostic*, not a *predictive*, equation] we can calculate the stream function—and hence the velocities—from that knowledge. Note that since (5.12) is a second-order, elliptic, equation, we need appropriate boundary conditions to determine the solution. So all the information of dynamical importance is implicit in the vorticity distribution; hence the importance of (5.10). In principle, then, (5.10) can be used to predict how the absolute vorticity distribution changes, then (5.9) tells us the vorticity distribution; then, assuming we know the boundary conditions, (5.12) can be solved for the stream function, and hence the velocity.

Note the analogy between (5.12) and the equation for electric potential V in the presence of a two-dimensional charge distribution $q(x, y)$; stream function ψ is analogous to potential V, vorticity ζ to charge q.

A concept related to vorticity is **circulation**. The circulation C around a closed contour $\mathcal{C}$ (see Fig. 5.5) is simply defined as

$$C = \oint_{\mathcal{C}} \mathbf{u} \cdot \mathbf{dl} \,, \tag{5.13}$$

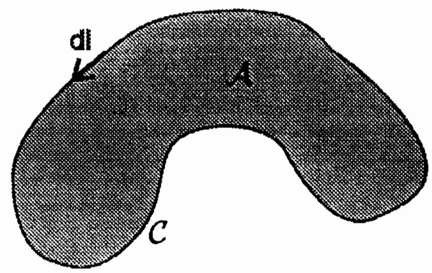

Figure 5.5: The contour C in the definition of circulation.

where the integral is around the contour and **dl** the linear increment along $\mathcal{C}$. But, from Stokes' theorem,

$$C = \oint_{\mathcal{C}} \mathbf{u} \cdot \mathbf{dl} = \int_{\mathcal{A}} (\nabla \times \mathbf{u}) \cdot \widehat{\mathbf{z}}\, dA \;=\; \int_{\mathcal{A}} \zeta\, dA\ , \tag{5.14}$$

where dA is the area element and $\mathcal{A}$ the area enclosed by $\mathcal{C}$. Thus, *the circulation around a closed contour is equal to the integrated vorticity enclosed by that contour.*

Example—the flow around a point vortex.

Suppose there is a **point vortex**, for which $\zeta(x, y) = Z_0 \delta(x - x_0)\delta(y - y_0)$ [so $\zeta = 0$ everywhere except at (x_0, y_0)]. Since we can anticipate the problem to have circular symmetry, we move into polar coordinates, with (x_0, y_0) as the origin (see Fig, 5.6). In polar coordinates, the Laplacian is

$$\nabla_2^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}$$

so that, if we look for symmetric solutions for which $\psi = \psi(r)$ then, everywhere except $r = 0$, $\nabla_2^2 \psi = 0$ or

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = 0\ .$$

The general solution is

$$\psi = A + B \ln r$$

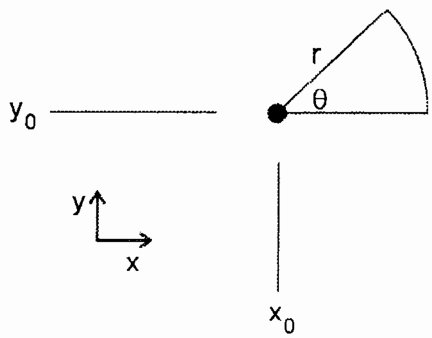

Figure 5.6: Geometry of point vortex example.

where A and B are constants. The constant A is irrelevant (as only the gradients of ψ have physical meaning); the velocity is

$$\mathbf{u} = -\widehat{\mathbf{z}} \times \nabla\psi = \left(\frac{\partial\psi}{\partial r}, -\frac{1}{r}\frac{\partial\psi}{\partial\vartheta}\right) = \left(\frac{B}{r}, 0\right) .$$

To determine B, we note that this circulation around any contour enclosing the point vortex is

$$C = \int \zeta \, dA = \iint Z_0 \delta(x - x_0)\delta(y - y_0) \, dx \, dy = Z_0 .$$

But if we choose a circular contour at radius r, then

$$C = \oint_C \mathbf{u} \cdot \mathbf{dl} = \int_0^{2\pi} u(r) \, r \, d\vartheta = 2\pi r \, u,$$

where u is the azimuthal velocity (see Fig 5.7), and so $B = Z_0/(2\pi)$. So the solution is

$$\begin{aligned} \psi(r) &= \frac{Z_0}{2\pi} \ln r ; \\ \mathbf{u} &= \left(\frac{Z_0}{2\pi r} , 0\right) . \end{aligned}$$

One important property of fluid flow—and rotating flow in particular—that this example makes clear is that the circulation is **nonlocal**: even a

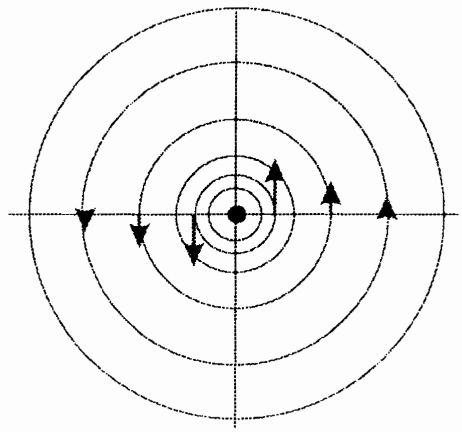

Figure 5.7: Circulation around a cyclonic point vortex (northern hemisphere).

localized vorticity will induce a *remote* circulation, just as electrical charges induce a remote field. Amongst other things, this means that one cannot in general think about fluid dynamics in terms of local, fluid parcel arguments, since the flow at the location of the parcel depends on the behavior of all other parcels.

5.4 Further reading

This material is covered in several geophysical fluid dynamics texts. The most suitable is Chapters 1 and 4 of:

"An Introduction to Dynamic Meteorology", J.R. Holton, Academic Press, 1979 (2nd edition).