12.215 Modern Navigation

Thomas Herring

Review of last class

- Sextant measurements using the sun:
  - We tracked the sun to find its highest elevation and the time this occurs.
- Our one cheat was using GPS to get time (and to check the results)

Today's Class

- Review of linear Algebra. Class will be based on the book "Linear Algebra, Geodesy, and GPS", G. Strang and K. Borre, Wellesley-Cambridge Press, Wellesley, MA, pp. 624, 1997
- Topics to be covered will be those later in the course
- General areas are:
  - Vectors and matrices
  - Solving linear equations
  - Vector Spaces
  - Eigenvectors and values
  - Rotation matrices

Basic vectors and matrices

- Important basic concepts
- Vectors: A column representing a set of n-quantities
  - In two and three dimensions these can be visualized as arrows between points with different coordinates with the vector itself usually having on end at the origin
  - The same concept can be applied to any n-dimensional vector
  - Vectors can be added and subtracted (head-to-tail) by adding and subtracting the individual components of the vectors.
  - Linear combinations of vectors can be formed by scaling and addition. The result is another vector e.g., c**v**+d**w**
  - (Often a bold symbol will be used to denote a vector and some times a line is drawn over the top).

Lengths and dot products

- The dot product or inner product of two vectors is defined as:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

- The order of the dot product makes no difference.
- The length or norm of a vector is the square-root of the dot product
- A unit vector is one with unit length
- If the dot product of two vectors is zero they are said to be orthogonal (orthonormal if they are unit vectors)
- The components of a 2-D unit vector are cos and sin of the angle the vector makes to the x-axis.

Angles between vectors

- The cosine formula:

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$$

- Schwarz inequality: The dot product of any two vectors is less or equal the product of the lengths of the two vectors
- The representation of a plane is by the vector that is normal to it. For a plane through the origin, all vectors in that plane must have zero dot product with the normal. This then provides an equation for the plane.

Planes

- If a plane does not contain then the origin, then the coordinates of a point on the normal containing the plane specifies the plane. The dot product of the normal and points in the plane then is a constant.
- If the normal to the plane is a expressed as a unit vector, the constant is the closest distance of the plane to the origin.
- In N-dimensional space, the concept of a plane is the same: it is defined by **n.v**=d
- Any two non-collinear vectors define a plane

Matrices and linear equations

- Any set of linear equations (i.e., equations which do not contain powers or products of the unknowns) can be written in matrix form with the coefficients of the linear equations being the elements of the matrix.
- The rows and columns of matrices are themselves vectors.
- A matrix represents a linear combination of the elements of a vector to form another vector of possibly different length:

$$\mathbf{Ax} = \mathbf{b} \text{ where } \mathbf{x} \text{ and } \mathbf{b} \text{ are vectors of length n and m}$$

$$\mathbf{A} \text{ is a m-rows and n column matrix}$$

Solving linear equations

- If the **x** and **b** vectors are the same length, then given **A** and **b** it is often possible to find **x** (sometimes this is not possible and sometimes **x** may not be unique).
- There a many methods for solving this type of system but the earliest ones are by elimination i.e., linear combinations are formed of the rows of the matrix **A** that eliminate one of the elements of **x**. The process is repeated until only one element of **x** remains (which is easily solved). Back substitution allows all the components of **x** to be computed.
- This process is sometimes viewed as multiplying by eliminator matrices.

Rules of matrix multiplication

- The product of two matrices **A** (n-rows and m-columns) by **B** (r-rows and c-columns) is only possible if m=r
- The resultant matrix has n-rows and c-columns.
- In general, **AB** does not equal **BA** even the matrices are square
- A matrix multiplication is the dot products of rows of the first matrix with the columns of the second matrix
- Matrix multiplication is associative (**AB**)**C**=**A**(**BC**) but not commutative
- A matrix is invertible if $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$ where **I** is a unit matrix, exists.

Factorization

- In factorization a matrix **A** is written as **A=LU** where is **L** is a lower triangular matrix and **U** is an upper triangular matrix.
- The individual matrices **L** and **U** are not unique (**L** can be multiplied by a scalar and **U** divided by the same scalar with out changing the product. Convention has the diagonal of **L** being 1's.
- Why factorize? Since forms are lower triangular, substitute down (L) and up (U) the matrix

$$\text{Solve } \mathbf{Lc} = \mathbf{b} \text{ then solve } \mathbf{Ux} = \mathbf{c} \text{ to solve } \mathbf{Ax} = \mathbf{b}$$

Characteristics of LU

- When the rows of A start with zero so do the corresponding rows of L; when the columns of A start with 0 so do the columns of U.
- Many estimation problems are “band-limited” i.e., only a small number of the elements around the diagonal are non-zero; the L and U matrices will also be band-limited but the inverse of such a matrix is normally full. (Factorization saves time and space).
- http://web.mit.edu/18.06/www/Course-Info/Mfiles/slu.m is a link to an SLU matlab code (also code at same site that pivots the matrix which is a more stable approach).

Transpose

- The transpose of a matrix is the matrix with rows and columns switched. Usually denoted as $\mathbf{A}^T$ or sometimes **A’**
- Some rules: $(\mathbf{AB})^T=\mathbf{B}^T\mathbf{A}^T$
- A symmetric matrix is one for which $\mathbf{A}=\mathbf{A}^T$
- The products $\mathbf{A}^T\mathbf{A}$ and $\mathbf{AA}^T$ generate symmetric matrices. We will see these forms many times in when we cover estimation and statistics.

Matrix rank

- The rank of a matrix is the number of non-zero pivots in the matrix. Pivots are the number you divide by when doing Gauss elimination or when solving the UL system.
- The rank is the number of independent rows in a matrix (i.e., rows that are not linear combinations of other rows).
- If the rank of a square matrix is less than its dimension then the matrix is singular: Two cases:
  - Infinite number of solutions
  - No solutions

Vector spaces

- An nxn matrix can be seen as defining a space made up of the n column vectors of the matrix
- Sub-spaces are vector spaces that satisfy two requirements: if **v** and **w** are vectors in a subspace and c is a scalar; the **v+w** is in the subspace and c**v** is in the subspace (eg., a plane is a subspace of 3-D space).
- Vectors that lie in a plane, all addition and scaling of these vectors lie in that same plane
- Non-singular matrices allow transformation back and forth between two spaces.

The null space

- An important case (especially in estimation) is the null space. This is the set of non-trivial vectors that satisfy the equation:

  $$\mathbf{Ax} = 0$$

- Any vectors **x** that satisfy this equation collapse to the null space meaning that you can reverse the transformation.
- In estimation, vectors that are in the null space can not be estimated and an arbitrary component of them can be added to any solution.
- Generally null spaces exist when the columns of a matrix are linearly dependent.

Eigenvectors and Eigenvalues

- This second applies to square matrices
- Normally when a matrix multiplies a vector, the vector changes direction (i.e., the dot product is not just the product of the lengths of the vectors)
- There are specific vectors that when multiplied by a specific matrix do not change direction.
- These are called eigenvectors and satisfy the equation: $\mathbf{Ax}=\lambda\mathbf{x}$
- By convention the eigenvectors are unit vectors.
- Normally the number of eigenvectors match the dimension of the matrix
- For symmetric matrices the eigenvectors are all orthogonal.

Equation for eigenvalues

- To solve the eigenvalue problem we find the values of λ that make the determinate of ($\mathbf{A}$-$\lambda\mathbf{I}$)=0
- Once a matrix is factored, the determinate is the sum of the diagonal of the U matrix.
- The determinate generates an n^{th} order polynomial for λ where n is the dimension of the matrix.
- The product of the eigenvalues is the determinate of the matrix
- The sum of the eigenvalues is the trace of the matrix (i.e., the sum of the diagonal elements).

Diagonalization of a matrix

- If the eigenvectors are put in a column matrix S, then

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

- Since the eigenvectors for a symmetric matrix are orthogonal, $\mathbf{S}^{-1}=\mathbf{S}^{T}$ (most commonly in estimation problems we encounter symmetric matrices.)
- Symmetric matrices with all positive eigenvalues are said to be positive definite.

Other matrices that we will encounter

- Rotation matrices: These are matrices that rotate a vector but do not change its size. They can be composed of three rotations about xyz axes: Form below is for a rigid body rotation. The signs are flipped for coordinate system rotations. The Z-rotation follows the pattern with the signs in the X position

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}; R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix};$$

Rotation matrices

- The form shown is for rotating a body, rotations of the coordinate system itself are of the same form except that the signs changes places.
- Note: When coordinate systems are rotated, after the first rotation the axes change location and hence the angle of the rotation about the new axes will change.
- In general, the order the rotation are applied will effect the results.
- If the direction cosines of a set of axes is known in the other frame, the rotation matrix can be directly written with the columns of the X system in the X' system

Direction cosines relationship

- Given two coordinate systems X and X', to find the coordinates of vector given in the X system in the X's a rotation matrix is constructed (could be made up of multiplication by the three basic rotation matrices)
- In general the rotation matrix looks like

$$\mathbf{X'} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \mathbf{X} \quad \leftarrow \text{Direction cosines of X' in X}$$

$$\uparrow \text{ Direction cosines of X in X' system}$$

Small angle rotations

- If the rotation are small, then cos(θ)~1 and sin(θ)~θ in this case the rotation matrix reduces to

$$R = \begin{bmatrix} 1 & -\theta_z & \theta_y \\ \theta_z & 1 & -\theta_x \\ -\theta_y & \theta_x & 1 \end{bmatrix}$$

where θ_x, θ_y, and θ_z are the rotations around each axis

- Rotation rate matrices are of this form (but after the rotation has persisted the axes change direction by a large and the equations needed to be integrated. (Example would be effects of gravitational torques on the equatorial bulge.

Summary

- General review of linear algebra:
  - Vectors and matrices
  - Solving linear equations
  - Vector Spaces
  - Eigenvectors and values
  - Rotation matrices
- These concepts will be used in latter classes when we cover satellite motions and estimation.