Lecture notes for 12.086/12.586, Modeling Environmental Complexity

D. H. Rothman, MIT
September 24, 2014

Contents

1 Anomalous diffusion

In our studies of river network geometry, we have seen how simple random walks create a hierarchy of shapes.

However the random walks need not be so simple, and generalizations of the diffusive scaling $\langle x^2 \rangle \sim t$ are possible.

Such behavior occurs when there is no characteristic jump size or when there is no characteristic waiting time between jumps.

The problem has lately attracted much controversy and attention in ecology (animal movement [1–3]) and sociology (human movement [4]).

It also occurs widely in physical problems of transport, such as in diffusion with traps, and has been applied with much success to problems of dispersion in groundwater flow [5]. Another example is diffusion in the presence of

convection rolls [6].

In this lecture, our principal goal is to illustrate how anomalous diffusion may arise, thereby suggesting one way in which deviations from the scaling predictions of simple random walks may arise in particular problems.

1.1 Beyond the central limit theorem

Our discussion of the random walk centered on analysis of the sum X_N of N random variables l_i:

$$X_N = \sum_1^N l_i$$

There we found that the distribution of X_N is Gaussian. Our sketch of the central limit theorem required that the second-moment $\langle l^2 \rangle$ be finite.

This condition is not met when the distribution $p(l)$ is "long-tailed" or "broad." Consider

$$p(l) \sim l^{-(1+\mu)}, \qquad \mu > 0, \qquad l \to \infty.$$

Assuming a lower limit l_0,

$$\langle \ell \rangle = \int_{l_0}^{\infty} l p(l) \mathrm{d}l = \int_{l_0}^{\infty} l^{-\mu} \mathrm{d}l$$

and

$$\langle \ell^2 \rangle = \int_{l_0}^{\infty} l^{1-\mu} \mathrm{d}l$$

Whether the mean or variance exists depends on the value of μ:

$$
\begin{aligned}
0 < \mu \leq 1 &: \quad \langle l \rangle, \langle l^2 \rangle \text{ infinite} \\
1 < \mu \leq 2 &: \quad \langle l \rangle \text{ finite}, \langle l^2 \rangle \text{ infinite} \\
\mu > 2 &: \quad \langle l \rangle, \langle l^2 \rangle \text{ finite}
\end{aligned}
$$

The latter case ($\mu > 2$) falls within the conditions of our derivation of the central limit theorem.

The others do not. A different limit theorem nevertheless exists [7,8] to show that the sum X_N converges to a "stable" distribution.

The resulting random walks are qualitatively different from those we've considered previously, and require a generalization of the diffusive law $\langle x^2 \rangle \propto t$.

This is called *anomalous diffusion*. It appears, at least phenomenologically, in varied settings: diffusion of passive scalars in turbulence, dispersion of contaminants in groundwater, animal foraging, human travel. The various empirical reports are however controversial.

Here we consider two related types:

- *Lévy flights.* Random walks with power-law step sizes at equal increments of time.

- *Continuous time random walk.* Random walks of equal step sizes but with power-law waiting times.

We proceed by examining the sum X_N.

1.2 Large fluctuations and scale invariance

First, we seek some statistical intuition and ask [8]: *What is the largest value $l_c(N)$ encountered among the N terms of the sum X_N?*

The probability that $l > l_c$ is chosen in a particular trial is

$$P(l > l_c) = \int_{l_c}^{\infty} p(l)\mathrm{d}l$$

If $l > l_c$ occurs at most once in N trials, we have, for large N,

$$P(l \leq l_c) \simeq 1, \qquad N \to \infty$$

The probability that l_c occurs only once in N trials is *maximized* when

$$N\,P(l > l_c)\,[P(l \leq l_c)]^{N-1} \simeq N\,P(l > l_c) = 1$$

Substituting $p(l) = l^{-(1+\mu)}$, we have

$$N\, l^{-\mu}\big|_{l_c(N)}^{\infty} \sim 1$$

and therefore

$$l_c(N) \sim N^{1/\mu}, \qquad N \to \infty.$$

For large but finite N, the region $l > l_c(N)$ is effectively unsampled.

For $0 < \mu \le 1$, we can therefore ignore the long tail of $p(l)$ and estimate the sum X_N by multiplying the "truncated" mean N times:

$$X_N \sim N \int^{l_c(N)} l p(l) \mathrm{d}l$$

We obtain

$$X_N \sim N \int^{N^{1/\mu}} l^{-\mu} \mathrm{d}l \sim \begin{cases} N(N^{1/\mu})^{(1-\mu)} = N^{1/\mu} & (\mu < 1) \\ N \ln N & (\mu = 1) \end{cases}$$

or, equivalently,

$$X_N \sim \begin{cases} l_c & (\mu < 1) \\ l_c \ln l_c & (\mu = 1) \end{cases}$$

Thus the typical value of the sum X_N is *dominated by the largest term* within it, $l_c(N)$.

Note that we have obtained this result even though $\langle l \rangle$ and therefore $\langle X_N \rangle$ do not exist.

We can obtain the variance by a similar agument. We have, for $\mu < 1$,

$$\begin{aligned} \mathrm{Var} X_N &= N \,\mathrm{Var}\, l \\ &= N\left(\langle l^2 \rangle - \langle l \rangle^2\right) \\ &\sim N\left[\left(N^{1/\mu}\right)^{(2-\mu)} - (X_N/N)^2\right] \\ &\sim N^{2/\mu} - N^{2/\mu - 1} \\ &\sim N^{2/\mu} \qquad (\mu < 1,\ N \to \infty) \end{aligned}$$

Here the limitation to $\mu < 1$ applies to the term $\langle l \rangle^2$. But when $\mu = 1$ its log divergence scales away nevertheless and we obtain the same result.

For $\mu > 1$ the mean
$$\langle l \rangle \sim \mathcal{O}\left(l_0^{1-\mu}\right) = \text{const.}$$

But $\langle l^2 \rangle$ still diverges for $\mu \leq 2$ so that

$$
\begin{aligned}
\mathrm{Var} X_N &= N\left(\langle l^2 \rangle - \langle l \rangle^2\right) \\
&\sim N^{2/\mu} - \text{const} \cdot N \\
&\sim N^{2/\mu} \qquad (\mu < 2, \ N \to \infty)
\end{aligned}
$$

where we obtain the same asymptotic scaling since $2/\mu > 1$.

Finally, note that the variance diverges logarithmically for $\mu = 2$. When $\mu > 2$, the variance grows linearly with N, as for a typical random walk, and the central-limit theorem applies.

Recalling that $l_c \sim N^{1/\mu}$, we thus have that the variance of the sum

$$\mathrm{Var} X_N \sim l_c^2, \qquad \mu < 2.$$

Conclusion: When $\mu < 2$, the sum X_N is dominated by the largest fluctuation $l_c(N)$. This holds true even for $1 < \mu < 2$ because the typical deviation of X_N from its mean scales like l_c.

1.3   Lévy flights

A picture of a Lévy flight looks like this: many small steps are occasionally followed by a much larger step, which are, collectively, followed by a much much larger step.

In this way there is no intrinsic scale to the process, and no length scale ever dominates.

In our present formulation, rms displacements r grow like

$$r \sim t^{1/\mu}.$$

This *super-diffusive* behavior for small $\mu < 2$ is a consequence of the instantaneous jump, i.e., assuming $t \propto N$.

Thus practical applications tend to include a modification for jump times, as we next consider.

1.4 Continuous time random walk

The CTRW is a random walk on a regular lattice with random waiting times τ between each step:

$$\psi(\tau) = \text{probability of waiting time } \tau.$$

It is essentially diffusion with "traps," in which the trap waiting time differs in both space and time.

As before, we take the jump size distribution to be $p(l)$, but with finite mean and variance.

After N steps, we have, as usual, the mean-square distance growing linearly with N:

$$\langle X_N^2 \rangle = \langle l^2 \rangle N, \qquad N \to \infty$$

where

$$\langle l^2 \rangle = \int l^2 p(l) \mathrm{d}l.$$

Nearest-neighbor hops $\Rightarrow \langle l^2 \rangle = $ the squared lattice spacing.

The total time t taken by the N hops is

$$t = \sum_{n=1}^{N} \tau_n.$$

The problem thus concerns the sum of random waiting times τ rather than random steps l.

If $\psi(\tau)$ is well behaved such that $\langle \tau \rangle$ is finite,

$$t \sim N \langle \tau \rangle$$

and

$$\langle X^2 \rangle = 2Dt, \qquad D = \frac{\langle l^2 \rangle}{2 \langle \tau \rangle},$$

i.e., typical diffusion.

The interesting case corresponds to a long-tailed $\psi(\tau)$ such that

$$\psi(\tau) \sim \tau^{-(1+\mu)}, \qquad 0 < \mu < 1, \qquad \tau \to \infty$$

When $\mu < 1$, $\langle \tau \rangle = \infty$. Applying the results of our earlier analysis, the total time t can be estimated from the largest expected waiting time τ_c in N steps:

$$t \sim N \int^{\tau_c} \tau \psi(\tau) \mathrm{d}\tau \sim N \int^{N^{1/\mu}} \tau^{-\mu} \mathrm{d}\tau \sim N^{1/\mu}, \qquad 0 < \mu < 1$$

After N steps of the random walk, we have, as usual

$$\langle X^2 \rangle = \langle l^2 \rangle N.$$

But now $N \sim t^\mu$ and therefore

$$\langle X^2 \rangle \sim \langle l^2 \rangle t^\mu, \qquad 0 < \mu < 1.$$

This behavior is called *sub-diffusive*, because $\langle X^2 \rangle$ grows slower than time $(0 < \mu < 1)$.

1.5   Diffusion on a comb

The most interesting cases of anomalous diffusion occur when the power-law distributions of step size or waiting times arise intrinsically, rather than by imposition.

Probably the simplest such case is a random walk on a comb.

We take the x-axis along the backbone of the comb and L as the length of each of the comb's equally spaced "teeth."

In considering diffusion along x, we must take account of the time spent "trapped" in the teeth.

1.5.1   Infinite L

For infinitely long teeth, $L \to \infty$, and the waiting time in a tooth is given by the distribution of first-passage times (already used in our study of river

basins):

$$\psi(\tau) \sim \tau^{-3/2}, \qquad \tau \to \infty$$

Applying the results of the previous section, we recognize $\mu = 1/2$ and conclude that

$$\langle x^2 \rangle \sim t^{1/2} \qquad \Rightarrow \qquad \langle x^2 \rangle^{1/2} \sim t^{1/4}$$

Thus the rms displacement grows subdiffusively, like $t^{1/4}$, rather than $t^{1/2}$.

1.5.2 Finite L

If the "trap" size L is finite, the time required to explore a given trap is

$$\tau_c \sim L^2/D_0,$$

where D_0 is the *bare diffusion coefficient* inside the trap.

For times $t \ll \tau_c$, the subdiffusive scaling of the previous section applies.

For $t \gg \tau_c$, however, we must consider that the average residence time in a trap is no longer infinite.

The distribution $\psi(\tau)$ instead has an effective cutoff such that

$$\langle \tau \rangle \sim \int^{\tau_c} \tau \, \psi(\tau) \, \mathrm{d}\tau = \int^{\tau_c} \tau^{-1/2} \mathrm{d}\tau = \tau_c^{1/2} \sim L.$$

Thus for times $t \gg \tau_c$, the effect of the traps is merely one of increasing the time interval between hops along the backbone x. That is, we have as usual

$$\langle x^2 \rangle = 2Dt, \qquad D = \frac{l^2}{2\langle \tau \rangle}, \qquad t \gg \tau_c$$

where $l =$ the tooth spacing. Substituting for $\langle \tau \rangle$, however, we find

$$D \sim 1/L.$$

Thus the effect of finite L is to scale the *effective diffusion coefficent* by $1/L$.

1.6   Lévy walks

The CTRW can be combined with Lévy flights to render the latter more physically appealing.

One idea is to combine the jump-size and waiting-time distributions into a joint probability density [9, 10]

$$\Psi(l,t) = p(l)\, \delta\left(t - \frac{l}{v(l)}\right)$$

where $p(l)$ has the usual power law tail and $v(l)$ is a (possibly) length-dependent velocity.

The resulting random walk is called a *Lévy walk* [9, 10]. It visits exactly the same sites as a Lévy flight, but waiting times scale with distance hopped.

Another idea is for the waiting times and step sizes to be independent [4], i.e.

$$\psi(\tau) = \tau^{-(1+\alpha)}, \qquad 0 < \alpha < 1$$

and

$$p(l) \sim l^{-(1+\beta)}, \qquad 0 < \beta < 2$$

where the restriction on α is such that the waiting times promote sub-diffusion whereas the jump sizes tend towards super-diffusion.

Applying our earlier results,

$$X \sim \langle l \rangle \sim N^{1/\beta}$$

and

$$t \sim \langle \tau \rangle \sim N^{1/\alpha}$$

and therefore

$$X \sim t^{\alpha/\beta}.$$

1.7   The lessons learned

- Long-tailed distributions break the central-limit theorem and produce large fluctuations, intermittency, and scale invariance; i.e., *variability*.

- These distributions need not be introduced *ad hoc.* Some problems, particularly transport in disordered media, generate them "for free."

- Resulting rms fluctuations $\left\langle x^2(t) \right\rangle^{1/2} \sim t^\alpha$ scale *anomalously,* with $\alpha \neq 1/2$.

References

[1] A. M. Edwards et al. Revisiting Lévy flight search patterns of wandering albatrosses, bumblebees and deer. *Nature* **449**, 1044–1049 (2007).

[2] Sims, D. W. *et al.* Scaling laws of marine predator search behaviour. *Nature* **451**, 1098–1102 (2008).

[3] Humphries, N. *et al.* Environmental context explains Lévy and Brownian movement patterns of marine predators. *Nature* **465**, 1066–1069 (2010).

[4] Brockmann, D., Hufnage, L. & Geisel, T. The scaling laws of human travel. *Nature* **439**, 462–465 (2006).

[5] Berkowitz, B. & Scher, H. Theory of anomalous chemical transport in random fracture networks. *Physical Review E* **57**, 5858 (1998).

[6] Young, W., Pumir, A. & Pomeau, Y. Anomalous diffusion of tracer in convection rolls. *Physics of Fluids A: Fluid Dynamics* **1**, 462 (1989).

[7] Gnedenko, B. V. & Kolmogorov, A. N. *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley, Reading, Massachusetts, 1968).

[8] Bouchaud, J.-P. & Georges, A. Anomalous diffusion in disordered media: statisical mechanisms, models, and physical applications. *Phys. Rep.* **195**, 127–293 (1990).

[9] Shlesinger, M. F., Zaslavsky, G. M. & Klafter, J. Strange kinetics. *Nature* **363**, 31–37 (1993).

[10] Shlesinger, M. F., Klafter, J. & Zumofen, G. Above, below and beyond Brownian motion. *American Journal of Physics* **67**, 1253–1259 (1999).

12.086 / 12.586 Modeling Environmental Complexity

Fall 2014