

## 12.009/18.352 Problem Set 5

Due Thursday April 9, 2015

Problem 1: 45 pts — (a,b,c,d)=(10,15,10,10)

Problem 2: 55 pts — (a,b,c,d,e,f)=(10,5,10,10,10,10)

1. *Borehole Paleothermometry*: One way we know about past climate variability is by drilling deep holes (boreholes) into bedrock or glaciers, and measuring the temperature at different depths. Surface temperature perturbations have diffused downwards below the surface, and we can solve for past surface temperatures by integrating the diffusion equation backwards in time, and solving for the surface boundary condition as a function of time. Here we will tackle a slightly easier example of how surface temperature variations can leave a direct temperature record below the ground or ice.

- (a) Assume that we have a large, homogeneous body of rock or glacial ice, infinite in horizontal extent, so that we can consider the time-dependent diffusion equation only in the vertical direction. Consider  $z$  to be the vertical coordinate, positive downwards. We will solve an equation for the temperature perturbation from the long-term mean,  $T'(z, t)$ . Let the surface boundary condition be:

$$T'(0, t) = \Delta \cos(\omega t), \quad (1)$$

i.e., oscillation about zero with an angular frequency  $\omega$  and an amplitude  $\Delta$ . Assuming a thermal diffusivity  $D$ , show that

$$T'(z, t) = \Delta \cos(\omega t - kz)e^{-kz} \quad (2)$$

represents a solution to the diffusion equation, if an appropriate value of  $k$  is used. What is the appropriate value of  $k$ ?

- (b) Discuss the form of the solution given by eqn. (2). You might find it helpful to make some plots in MATLAB. (Hint: how do the depths of locations with  $T' = 0$  vary in time?)
  - (c) Using a value of  $\omega$  appropriate for the annual cycle, a temperature amplitude  $\Delta$  of 20°C, and a thermal diffusivity of  $10^{-6}$  m<sup>2</sup>/s, how deep must you go to get to the point where the amplitude of the seasonal cycle has decayed to 1°C? What is the temperature anomaly at that depth when it is summer at the surface, and the temperature is maximum there?
  - (d) How deep does a borehole have to be to observe a full cycle of a hypothesized mode of oscillatory climate variability with timescale  $\tau$ ? Discuss some practical limits of the application of this method of climate measurement.
2. *Precession of the Major Axis*: Without any perturbations from the outside, a planet orbiting in an elliptical orbit will remain in the same orbit forever. It turns out, however, that the stability of these fixed ellipses is tied very closely to the  $1/r$  potential in which the planets sit. In this problem, we will explore what happens if we add a

small perturbing force,

$$f \sim \frac{1}{r^4}, \quad (3)$$

to our system. This higher order term comes about both from interactions with other planets and through relativistic considerations – here we will quote the relativistic results alone for the strength of the perturbing force, since perturbations from other planets are complicated to calculate, and introduce forces that scale with other powers of  $r$  (e.g.  $f \sim r, f \sim 1/r^3$ ) – it is not immediately clear what terms dominate on long time scales. We will start with a conversion of our orbital equation from one in terms of the potential, to one in terms of the force. Based on the equation for the orbital energy derived in class, it can be shown that an expression can be written in terms of the force field,  $F(r)$ :

$$\frac{d^2}{d\theta^2} \left( \frac{1}{r} \right) + \frac{1}{r} = -\frac{\mu r^2}{l^2} F(r). \quad (4)$$

Showing this is a little involved, so we're not asking that you do it, but eqn. (4) will be the basis for what follows.

(a) By enforcing the gravitational potential and including the following definitions:

$$\frac{1}{\alpha} \equiv \frac{Gm_1^2 m_2}{l^2}, \quad u = \frac{1}{r} \quad (5)$$

show that equation (4) reduces to

$$\frac{d^2 u}{d\theta^2} + u = \frac{1}{\alpha}. \quad (6)$$

(b) Show that the conic section solution for the gravitational potential,

$$u = \frac{1}{\alpha} (1 + \epsilon \cos(\theta)), \quad (7)$$

is a solution to equation (6).

(c) At this point, we are ready to explore the consequences of including a small term which depends on  $r^{-4}$  to  $F(r)$ . This allows us to alter equation (6) to get

$$\frac{d^2 u}{d\theta^2} + u = \frac{1}{\alpha} + \delta u^2 \quad (8)$$

where  $\delta$  is a very small constant. Specifically,  $\delta$  must be small relative to the orbital radius; relativity predicts  $\delta = 3Gm_2/c^2 \approx 4400$  m, which is indeed miniscule compared to Earth's orbital radius of  $1.5 \times 10^{11}$  m. Because the  $u^2\delta$  term is so small it can be solved through a series of successive guesses where we start by guessing the solution of equation (6) and plug it into equation (8). By this we mean that we guess that  $u \approx u_0 + u_1$ , where  $u_1 \ll u_0$ , and then determine  $u_0$  as

the solution to the unperturbed equation:

$$\frac{d^2 u_0}{d\theta^2} + u_0 = \frac{1}{\alpha} \quad (9)$$

The perturbation  $u_1$  is solved for by using  $u^2 \approx u_0^2$  and then solving for the LHS that matches  $\delta u_0^2$ :

$$\frac{d^2 u_1}{d\theta^2} + u_1 = \delta u_0^2. \quad (10)$$

Since  $\delta$  is very small, we expect the correction  $u_1$  to be very small, and thus terms like  $u_0 u_1$  in the square of  $u$  are further small corrections to the solution, which we will not consider. For this problem, use  $u_0$  given by (7), and show that:

$$u_1 = \frac{\delta}{\alpha^2} \left( 1 + \frac{\epsilon^2}{2} + \epsilon \theta \sin \theta - \frac{\epsilon^2}{6} \cos 2\theta \right) \quad (11)$$

is a solution consistent with (10).

- (d) Show, remembering the small angle approximation, that our solution for  $u_0 + u_1$  can be simplified to

$$u \cong \frac{1}{\alpha} \left[ 1 + \epsilon \cos \left( \theta - \frac{\delta}{\alpha} \theta \right) \right] + \frac{\delta}{\alpha^2} \left( 1 + \frac{\epsilon^2}{2} - \frac{\epsilon^2}{6} \cos 2\theta \right). \quad (12)$$

- (e) Explain how the  $-(\delta/\alpha)\theta$  term contributes to the precession of the major axis of an elliptical orbit. You may find it helpful to make a few plots, using MATLAB, of  $r(\theta)$  for varying values of  $\delta$ ,  $\alpha$  and  $\epsilon$ . To start off with, try  $\delta = \pi/40$ ,  $\alpha = 1$  and  $\epsilon = 0.5$ . You might find `polar` a useful MATLAB function.
- (f) Estimate the timescale over which the effects of nonzero  $\delta$  could be noticed for the Earth. Do you think we can measure this predicted effect of relativity for any of the planets in the solar system?

MIT OpenCourseWare  
<http://ocw.mit.edu>

12.009J / 18.352J Theoretical Environmental Analysis  
Spring 2015

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.