12.009/18.352 Problem Set 2

Due Thursday, 26 February 2015

100 points total

Problem 1: 15 pts — (a,b)=(10,5)

Problem 2: 45 pts — (a,b,c,d,e,f)=(5,5,5,10,10,10)

Problem 3: 40 pts — (a,b,c,d,e,f)=(5,5,5,5,10,10)

1. *Kelvin's age of the Earth.* In this problem, we'll repeat the initial wrongheaded calculation by Lord Kelvin of Earth's age. The value here will differ somewhat from the 20 Myr quoted in the course notes, which was a later revision from his initial estimate.

   Assume the Earth started out as a homogeneous molten sphere, which uniformly solidified at temperature T_M, at a time $\tau_\oplus$ before present. Also assume a known present near-surface geothermal gradient $-\partial T/\partial z|_0$, a constant thermal diffusivity D, and constant surface temperature boundary condition T_0.

   Kelvin's values for these parameters are $T_M - T_0 = 7000°\text{F}$, $-\partial T/\partial z|_0 = 1°\text{F}/50\text{ft}$, and $D = 400\text{ft}^2/\text{yr}$.

   (a) Justify the assumption that the spherical geometry of Earth, and the fact that it is not semi-infinite in spatial extent, can be ignored for the purposes of this problem.

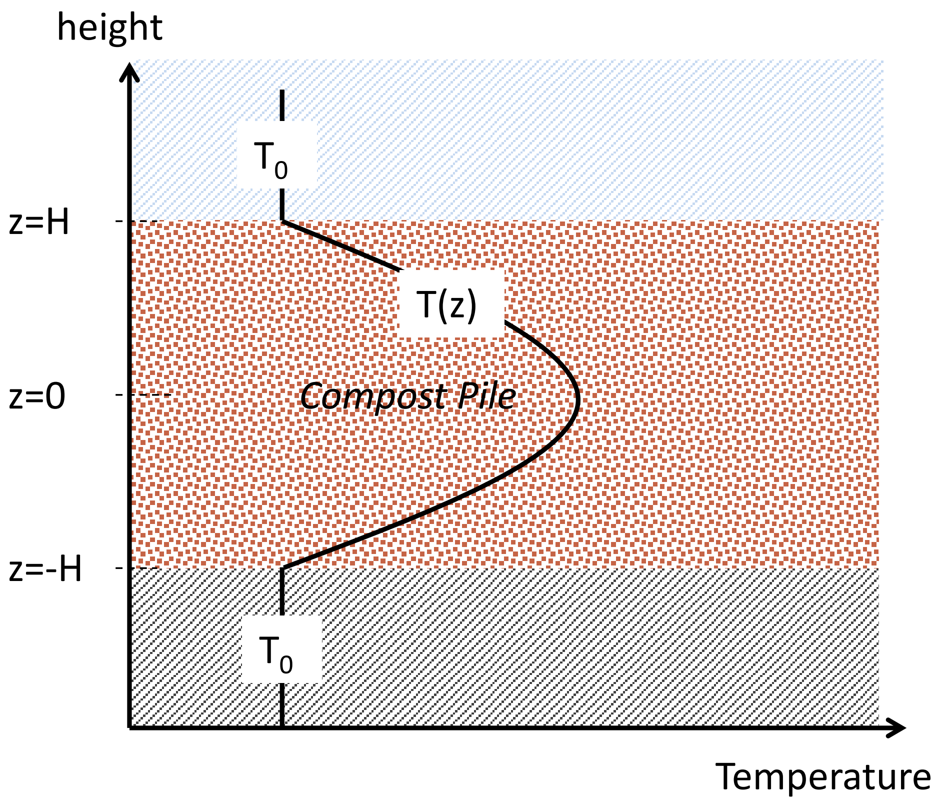
   (b) Derive a general expression for $\tau_\oplus$ and put in parameters to arrive at Kelvin's estimate of $\tau_\oplus^{\text{Kelvin}}$.

2. *Thermal Diffusion in a Compost Pile.* In this problem we will discuss the steady temperature profile of an idealized 1D (height-only) compost pile. A compost pile allows for rapid decomposition of organic matter because the temperature inside the pile can be greatly elevated above the temperature of the pile's surroundings, and decomposition (which supplies heat to the pile) increases as the temperature increases. Thus, there is a positive feedback loop between the decomposition rate and the temperature of the pile, which in some cases can be strong enough to heat a pile to the point of combustion!

   (a) Consider a pile confined between $z = -H$ and $z = H$, with upper and lower boundaries fixed to a temperature T_0 (Figure 2a). We will assume that all heat is lost due to thermal diffusion, and we will initially consider a pile where there is no temperature feedback, so decomposition generates heat at a constant volumetric rate Q_0 (units: W m^{-3}) throughout the pile. The compost has thermal diffusivity D (units: m^2 s^{-1}), and volumetric heat capacity C_p (units: J m^{-3} K^{-1}). Thus, the steady-state diffusion equation for the temperature of the pile is:

   $$D\frac{d^2T}{dz^2} = -\frac{Q_0}{C_p}. \tag{1}$$

   Note that this is now an ordinary differential equation (ODE) – since there is no time-dependence, T is a function of z alone. Solve for the profile of temperature perturbation from the background, $T'(z) \equiv T(z) - T_0$ in the compost pile.



(b) From now on, we'll drop the prime (on T') and consider T to represent the perturbation from the background temperature. Consider the somewhat harder problem where the decomposition rate scales linearly with temperature:

$$D\frac{d^2T}{dz^2} = -\frac{Q_0}{C_p}[1 + \alpha T]. \tag{2}$$

Here, we have introduced a new parameter, α, which represents the fractional change in decomposition rate for a 1-degree change in temperature (note that in reality, the functional dependence of biological activity on temperature is better modeled as exponential but this would make the problem a nonlinear ODE – still analytically solvable, but nastier). The toy problem given by equation (2) is a linear, second-order, nonhomogeneous ODE with constant coefficients – quite tractable. We can write $T = T_h + T_p$, where T_h is the *homogeneous solution*, and satisfies:

$$D\frac{d^2T_h}{dz^2} + \frac{\alpha Q_0}{C_p}T_h = 0. \tag{3}$$

What is the general form of T_h? What is the *particular solution* T_p?

(c) Using the boundary conditions that $T = T_h + T_p = 0$ on $z = (-H, H)$, solve for the coefficients in T_h and thus for the overall solution to $T(z)$. Discuss the relevance of the length scale $L \equiv \sqrt{\frac{DC_p}{\alpha Q_0}}$.

(d) *Real-world composting.* Let's plug in some values and see where it gets us. Assume $D = 2 \times 10^{-7}$ m^2 s^{-1}, $Q_0 = 10$ W m^{-3}, and $C_p = 10^6$ J m^{-3} K^{-1}, and pick a value of α that seems reasonable to you. What is the length scale L? Using MATLAB plot the temperature profiles from your solutions to parts 2a and 2c for a few values of H/L and discuss the importance of the temperature-dependence of decomposition for your results.

(e) OK, now for a question that should always be asked of simplified mathematical models of real-world phenomena – does your answer make physical sense? One way to think about this is in terms of comparing your solution from part 2c to your solution from part 2a. Do the solutions converge in the limit that the temperature sensitivity of the decomposition rate becomes very small, $\alpha \to 0$? Why or why not? Use a combination of graphical, mathematical, and physical reasoning to explain your results. If you want to take a formal limit, you should evaluate the limit at constant H (not constant H/L).

(f) Another way to think about whether your answer makes physical sense is in terms of the approach to equilibrium – starting from a state of zero temperature perturbation, the pile should heat up with time until it reaches equilibrium. What constraints (if any) does this pose on the range of H/L for which the solution is physical? If there is some range of H/L for which your solution is unphysical, what is going on?

3. *Growth of a Cloud Droplet.* How rapidly can a cloud droplet grow by condensation from supersaturated air? This question presents a highly idealized model of the problem, based again on the steady-state diffusion equation. This time we will work in spherical coordinates, and we will assume that the water vapor concentration q (units: kg m^{-3}) depends only on the radius from the center of the drop, r.

(a) Show that at steady state the diffusion equation for q becomes:

$$D_v \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dq}{dr}\right) = 0, \tag{4}$$

where D_v is the molecular diffusivity of water vapor in air.

(b) If we impose the boundary conditions that $q = Hq^*$ at $r = \infty$ and $q = q^*$ at $r = a$, where a is the radius of the droplet, then what is the solution to equation (4)? Here, q^* represents the water vapor mass concentration of a parcel of air that is saturated, and H is the relative humidity (humidity relative to saturation). $H > 1$ (greater than 100 %) indicates supersaturation of the air around the droplet and allows for growth with time, while $H < 1$ implies subsaturated air and allows for evaporation of the droplet. Show that the diffusive flux of water vapor into the sphere (J_v), in mass per unit time, is given by:

$$J_v = 4\pi D_v a q^*(H-1). \tag{5}$$

(c) Now, we want to consider how the droplet grows in time, due to the diffusive flux of water vapor into the droplet. Note that if we want to use equation (5), we need to justify our assumption that the atmospheric water vapor field is quasi-steady. In other words, we need to justify our neglect of the time-derivative of the water vapor concentration in our derivation up to this point. How might we do this?

(d) Assuming that we can use equation (5) to model the growth rate of a cloud droplet, write down the relevant differential equation for the time rate of change of a droplet radius a, given condensation rate J_v, and assuming that the droplet density is that of water, ρ_w.

(e) Solve the differential equation for droplet growth rate, subject to the initial condition that $a = a_0$ at $t = 0$. How does the radius depend on time? Is this surprising?

(f) Using $D_v = 2 \times 10^{-5}$ m^2 s^{-1}, $q^* = 5 \times 10^{-3}$ kg m^{-3}, and $H = 1.005$ (clouds are normally only very weakly supersaturated), make a log-log plot of $a(t)$ for droplets with a_0 =1, 10, and 100 μm. How long does it take a cloud droplet to grow to 1 mm under these conditions? If these conditions are relatively favorable for droplet growth, and a cloud droplet needs to grow to $\sim$1 mm before it becomes a raindrop, what does that tell you about mechanisms for raindrop formation?