12.005 Lecture Notes 4

Stress Tensor (continued)

Before getting bogged down in rotation and notation, let's consider a special case: rotation of the coordinate system by 45° about x_3.
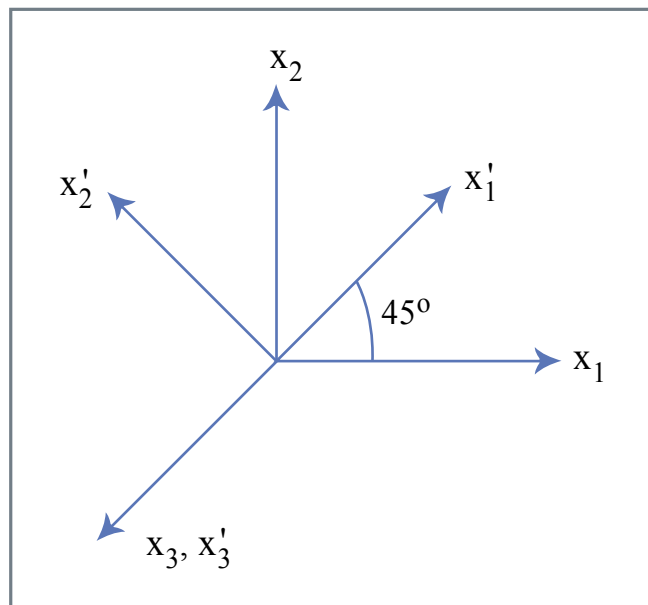

Figure 4.1
Figure by MIT OCW.

In parallel, consider the tractions on a plane with its normal in the $x_1 - x_2$ plane at 45° to x_1 and x_2.

$$\hat{n} = (\sqrt{2}/2, \sqrt{2}/2, 0)^{T} = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{T} = \underset{\sim}{\sigma} \cdot \underset{\sim}{n} = \frac{\sqrt{2}}{2} \begin{pmatrix} \sigma_{11} + \sigma_{12} \\ \sigma_{12} + \sigma_{22} \\ \sigma_{13} + \sigma_{23} \end{pmatrix}$$
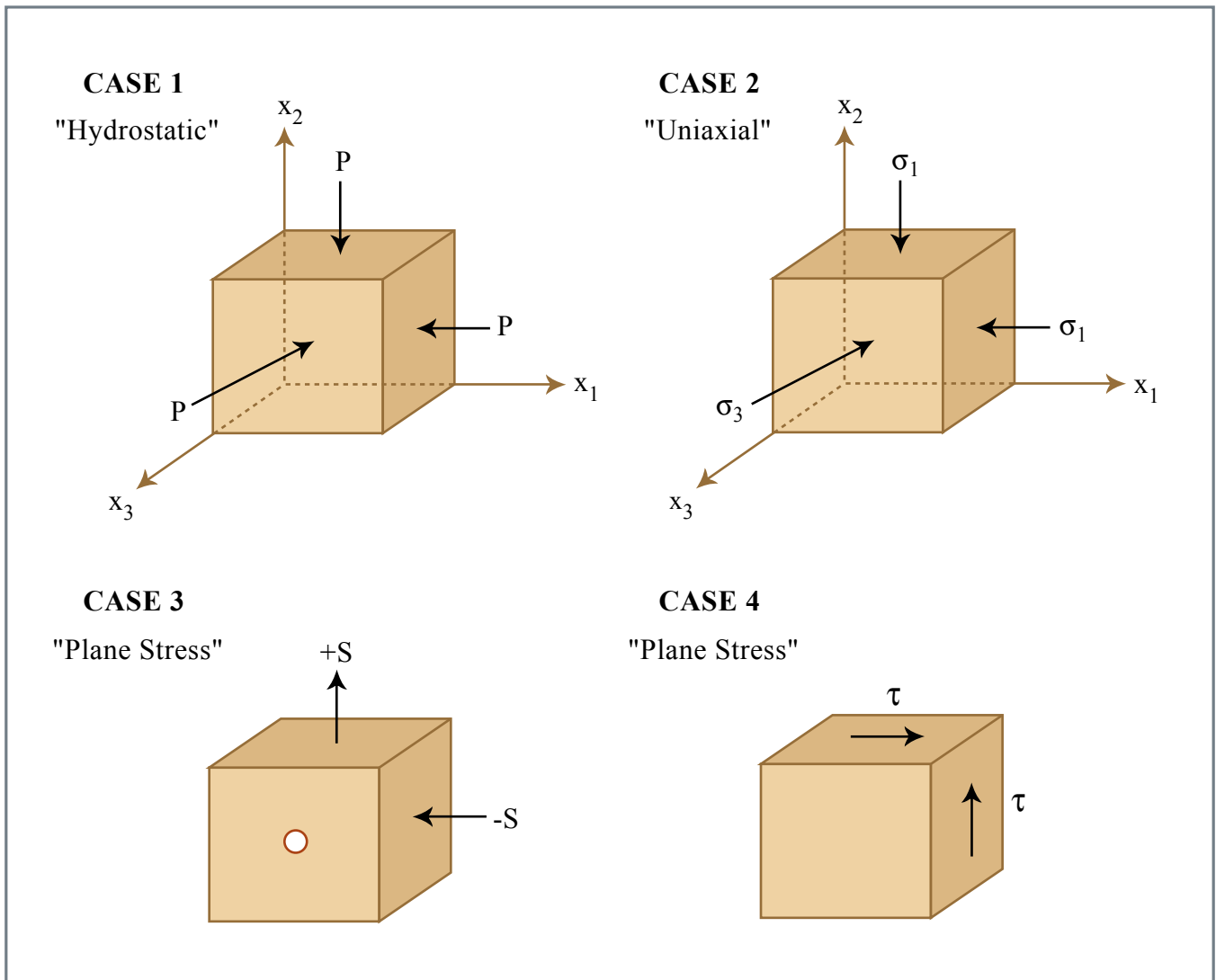
Figure 4.2

Figure by MIT OCW.

Notes:

Case 1: $\underset{\sim}{T}$ (on plane at 45°) is only $\sigma_n = p$, no τ.

Case 2: $\underset{\sim}{T}_{45°}$ is still only $\sigma_n = \sigma_1$.

Case 3: $\underset{\sim}{T}$ on faces 1 and 2 only σ_n, no τ.

   $\underset{\sim}{T}_{45°}$ is pure τ, no σ_n.

Case 4: $\underset{\sim}{T}$ on faces 1 and 2 all τ, no σ_n.

   $\underset{\sim}{T}_{45°}$ is all σ_n, no τ.

Evidently, there are "special" directions, at least for these cases, where $\tau = 0$!

Is this true in general? Can we find the "principal" frame where

$$\sigma_{ij}{}^{P} = \begin{pmatrix} \sigma_{11}{}^{P} & 0 & 0 \\ 0 & \sigma_{22}{}^{P} & 0 \\ 0 & 0 & \sigma_{33}{}^{P} \end{pmatrix} = \begin{pmatrix} \sigma_{1} & 0 & 0 \\ 0 & \sigma_{2} & 0 \\ 0 & 0 & \sigma_{3} \end{pmatrix} \quad ?$$

Require $\underset{\sim}{T}$ is parallel to $\hat{n}$ or

$$\begin{pmatrix} T \end{pmatrix} = \begin{pmatrix} \underset{\approx}{\sigma} \end{pmatrix} \begin{pmatrix} n \end{pmatrix} = \sigma \begin{pmatrix} n \end{pmatrix}$$

where $\underset{\approx}{\sigma}$ is tensor and σ is scalar (sometimes λ),

or

$$\begin{bmatrix} \begin{pmatrix} \underset{\approx}{\sigma} \end{pmatrix} - \sigma \underset{\approx}{I} \end{bmatrix} \begin{pmatrix} n \end{pmatrix} = 0$$

The stress tensor can be represented in different ways to highlight particular features or aid in solving geodynamic problems. This lecture explores how to represent the stress tensor in terms of principle stresses and isotropic and deviatoric stresses. Representing the stress tensor in terms of principle stresses makes visualizing the state of stress easier because it reduces the stress tensor to only three numbers. It also makes some

calculations easier.   Representing the stress tensor in terms of isotropic and deviatoric

stresses is helpful in determining the type of faulting produced by certain stresses.

Principle Stresses and Principle Axes

The stress tensor is a matrix that specifies the tractions on three mutually perpendicular

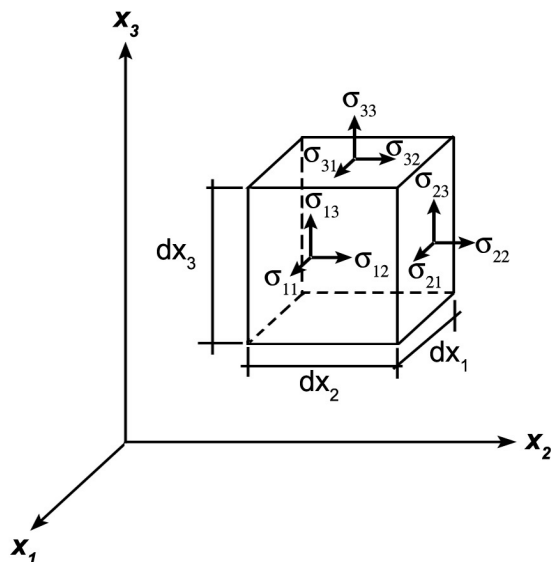faces of an infinitesimal cube.

Figure 4.3

Figure by MIT OCW.

In general, these tractions are both parallel and perpendicular to the normal vectors of the

faces. At a certain orientation of the faces, however, the tractions are only parallel to the

normal vectors. The directions of these normal vectors are called principle directions and

the stresses are called principle stresses. Calculating the principle directions and stresses

requires using eigenvalues.

1. Begin with Cauchy's formula:

$$T_i = \sigma_{ji} n_j$$

Require the traction T_i to be parallel to the normal vector by writing

$$T_i = \lambda n_i$$

in which λ represents one of the three principle stresses.

2. Combine the above equations:

$$\sigma_{ji} n_j = \lambda n_i$$

$$(\sigma_{ij} - \lambda \delta_{ij}) n_j = 0$$

$$\begin{bmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

In the middle equation, δ_{ij} is the Kronecker delta. It equals 0 if i≠j and equals 1 if i=j.

3. Solve:

The matrix equation has a solution if

$$\begin{Vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{Vmatrix} = 0$$

where the double bars indicate the determinant. This results in a third-degree equation in λ

$$-\lambda^3 + I_1\lambda^2 + I_2\lambda + I_3 = 0$$

where

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$-I_2 = \begin{Vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{Vmatrix} + \begin{Vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{Vmatrix} + \begin{Vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{Vmatrix}$$

$$I_3 = \begin{Vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{Vmatrix}$$

Isotropic and Deviatoric Stress

The stress tensor can be divided into two parts: isotropic stress and deviatoric stress. The isotropic stress $\sigma_{ij}^{\,0}$ is defined as

$$\sigma_{ij}^{\,0} = \frac{1}{3}\sigma_{kk}\delta_{ij}$$

It represents the mean normal stress or pressure. Subtracting the mean normal stress from the stress tensor produces the deviatoric stress $\sigma_{ij}^{\,d}$

$$\sigma_{ij}^{\,d} = \sigma_{ij} - \sigma_{ij}^{\,0}$$

The deviatoric stress represents the part of the stress that differs from a hydrostatic state.