Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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7.4 Steady onshore wind in a shallow Sea

Let us model the effect of turbulence by a constant eddy viscosity. Assume that convective inertia is negligible, the seabed is horizontal and vertical shear is important, the governing equation in a shallow sea are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{7.4.1}$$

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} \tag{7.4.2}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial z} + \nu\frac{\partial^2 u}{\partial z^2} \tag{7.4.3}$$

The boundary conditons are

$$u = v = w = 0, \quad z = -h \tag{7.4.4}$$

$$\mu\frac{\partial u}{\partial z} = \tau_x^S, \quad \mu\frac{\partial v}{\partial z} = \tau_y^S \tag{7.4.5}$$

As in a thin boundary layer, the vertical shear dominates.

Integrating over depth and defining the horizontal transport rates,

$$U = \int_{-h}^{0} u dz, \qquad V = \int_{-h}^{0} v dz \tag{7.4.6}$$

then the

$$\begin{aligned}
\frac{\partial \eta}{\partial t} &+ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \\
\frac{\partial U}{\partial t} &- fV = -gH\frac{\partial \eta}{\partial x} + \frac{\tau_x^S}{\rho} - \frac{\tau_x^B}{\rho} \\
\frac{\partial V}{\partial t} &+ fU = -gH\frac{\partial \eta}{\partial y} + \frac{\tau_y^S}{\rho} - \frac{\tau_y^B}{\rho}.
\end{aligned}$$

where

$$\tau_x^S = \rho\nu\left[\frac{\partial u}{\partial z}\right]_{z=0}, \quad \tau_y^S = \rho\nu\left[\frac{\partial v}{\partial z}\right]_{z=0} \tag{7.4.7}$$

$$\tau_x^B = \rho\nu\left[\frac{\partial u}{\partial z}\right]_{z=-h}, \quad \tau_y^B = \rho\nu\left[\frac{\partial v}{\partial z}\right]_{z=-h} \tag{7.4.8}$$

As an order estimate

$$U \sim \frac{\tau^S}{\rho f} \sim \frac{u_*^2}{f}$$

where u_* is the friction velocity. A vertical boundary layer (of Ekman) can exist wherein Coriolis force $\rho f U$ balances the viscous stress

$$\tau^B \sim \rho\nu \frac{\partial u}{\partial z} \sim \rho\nu \frac{U}{h}\frac{1}{\delta}$$

Therefore, the Ekman layer thickness is

$$\delta = O\left(\sqrt{\frac{\nu}{f}}\right)$$

A typical value of eddy viscosity is $\nu = 1\ cm^2/s$ and $f = 10^{-4}\ 1/s$. Therefore the Ekman layer thicknss is $O(1)$ m.

Our strategy is to get the horizontal transport, then the details of the boundary layers.

7.4.1 Wind setup due to steady onshore wind

Consider an infinitely long coastline along the x axis. Assume τ_y^S to be a given constant. Consider the steady state $\partial/\partial t = 0$ and ignore τ^B first. Beginning from the equations :

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0.$$

$$\begin{aligned} -fV &= -gH\frac{\partial \eta}{\partial x} \\ +fU &= -gH\frac{\partial \eta}{\partial y} + \frac{\tau_y^S}{\rho} \end{aligned}$$

with

$$V = 0, \quad y = 0 \tag{7.4.9}$$

Because of the infinite coast,

$$U = \frac{\partial \eta}{\partial x} = 0,$$

we must have

$$\frac{\partial V}{\partial y} = 0,$$

It follows that $V = 0$ everywhere, and

$$gH\frac{\partial \eta}{\partial y} = \frac{\tau_y^S}{\rho}. \tag{7.4.10}$$

meaning that there is a sea-level **Set-up**.

$$\begin{aligned}\eta &= \frac{\tau_y^S}{gH}\,y + \text{ constant}\\ &= \frac{\rho u_*^2}{\rho g H}\,y = \frac{u_*^2}{gH}\,y.\end{aligned}$$

If $u_* = 1$ cm/sec then $\tau_y^S = 0.1$ Pa. If $g = 10$ m/sec^2 and $H = 30$ m

$$\frac{u_*^2}{gH} = \frac{(10^{-2})^2}{10 \cdot 30} = 3 \times 10^{-7}.$$

Note that

$$1\,\text{atm} = 10^5 N/m^2, \quad 1N/m^2 = 1Pa = \frac{1}{670}\,psi.$$

For $g = 10m/s^2$ $\quad H = 30m$ the set up is calculated as follows.

τ_y^S	$\frac{\partial\eta}{\partial y}$	Δy	$\Delta\eta$
0.1 Pa	3×10^{-7}	100 km	3 cm
3 Pa		300 km	3 m

Although there is no mean flow (or flux), there is internal flow. We now look at the detailed distribution in z, by deviding the depth into three parts: the geostrophic interior, the surface Ekman layer, and the bottom Ekman layer.

7.4.2 Geostrophic core

Outside the boundary layers, we have

$$-fv_g = -g\frac{\partial\eta}{\partial x} = 0, \tag{7.4.11}$$

$$fu_g = -g\frac{\partial\eta}{\partial y} = -\frac{u_*^2}{H} \tag{7.4.12}$$

In this geostropic balance, there is a longshore current in the core.

7.4.3 Surface Ekman layer

Now viscosity is important, so that

$$\begin{aligned}-fv &= -g\,\frac{\partial\eta}{\partial x} + \nu\,\frac{\partial^2 u}{\partial z^2}\\ fu &= -g\,\frac{\partial\eta}{\partial y} + \nu\,\frac{\partial^2 v}{\partial z^2}.\end{aligned}$$

For this example

$$\frac{\partial \eta}{\partial x} = 0, \quad \frac{\partial \eta}{\partial y} = \frac{\tau_y^S}{\rho g H} = \frac{\rho u_*^2}{\rho g H} = \frac{u_*^2}{gH}$$

hence

$$-fv = \nu \frac{\partial^2 u}{\partial z^2} \tag{7.4.13}$$

$$fu = -\frac{u_*^2}{H} + \nu \frac{\partial^2 v}{\partial z^2}. \tag{7.4.14}$$

As long as $\delta/H \ll 1$, bounday-layer approximation can be made. Let us make some estimates based on empirical data, cited from Csanady :

$$\delta = 0.1 \frac{u_*}{f}, \quad \nu = \frac{u_*^2}{200f}, \quad Re_* = \frac{u_* \delta}{\nu} = 20$$

Pedlosky :

$$\nu = 1 \sim 10^3 cm^2/sec$$

$$\delta = \sqrt{\frac{1 \sim 10^3}{10^{-4}}} cm = 10^2 cm \sim 3 \times 10^3 cm.$$

Let the total velocity in the surface boundary layer be

$$u = u_g + u_E = -\frac{u^{*2}}{fH} + u_E, \quad v = v_g + v_E = v_E. \tag{7.4.15}$$

so that (u_E, v_E) are the boundary layer corections. Then for $z < 0$, we have

$$-fv_E = \nu \frac{\partial^2 u_E}{\partial z^2} \tag{7.4.16}$$

$$fu_E = \nu \frac{\partial^2 v_E}{\partial z^2} \tag{7.4.17}$$

The boundary conditions are

$$\nu \frac{\partial u_E}{\partial z} = 0, \qquad \nu \frac{\partial v_E}{\partial z} = u_*^2 \qquad \text{on } z = 0$$

$$u_E, \ v_E \to 0 \qquad z \to -\infty.$$

This is the Ekman boundary-layer problem. The solution is best obtained by introducing the complex velocity,

$$q_E = u_E + iv_E$$

then

$$q_E = \nu \frac{\partial^2 q_E}{\partial z^2}$$

$$\text{or} \quad \frac{d^2 q_E}{dz^2} - \frac{if}{\nu}\, q_E = 0$$

Let the solution be of the form,

$$q_E \propto e^{Dz}$$

then

$$D^2 - \frac{if}{\nu} = 0$$

Since

$$(i)^{1/2} = \pm e^{i\pi/4} = \pm\frac{1+i}{\sqrt{2}}.$$

We get

$$q_E = A \exp\left(\frac{1+i}{\sqrt{2}}\, \frac{z}{\sqrt{\nu/f}}\right) = A\, e^{(1+i)^{z/\delta}}. \tag{7.4.18}$$

Let

$$\delta = \sqrt{\frac{2\nu}{f}} \tag{7.4.19}$$

denote the boundary layer thickness. Apply the boundary condition on the sea surface,

$$\nu\, \left.\frac{\partial q_E}{\partial z}\right|_0 = i\, u_*^2$$

hence

$$A = \frac{i\, u_*^2\, \delta}{(1+i)\nu}.$$

The solution is

$$q_E = u_E + i\, v_E = \frac{i\delta\, u_*^2}{(1+i)\nu}\, e^{(1+i)z/\delta} = \frac{\delta\, {u^*}^2}{2\nu}(1+i)\, e^{z/\delta}\, \left(\cos\,\frac{z}{\delta} + i\,\sin\,\frac{z}{\delta}\right) \tag{7.4.20}$$

Separating real and imaginary parts, we get the velocity components,

$$u_E = \frac{\delta}{2}\,\frac{u_*^2}{\nu}\, e^{z/\delta}\, \left[\cos\,\frac{z}{\delta} - \sin\,\frac{z}{\delta}\right] \tag{7.4.21}$$

$$v_E = \frac{\delta}{2}\,\frac{u_*^2}{\nu}\, e^{z/\delta}\, \left[\cos\,\frac{z}{\delta} + \sin\,\frac{z}{\delta}\right]. \tag{7.4.22}$$

The hodograph is shown in Figure 7.4.1,

1. Physical Remark #1:
   Maximum velocity occurs on $z = 0$:

$$\frac{u_E(0)}{u_*} = \frac{v_E(0)}{u_*} = \frac{u_*}{f\delta}\left(\gg \frac{U}{u_* h} = \frac{u_*}{fh}\right).$$

   and is 45 degrees to the right of wind.

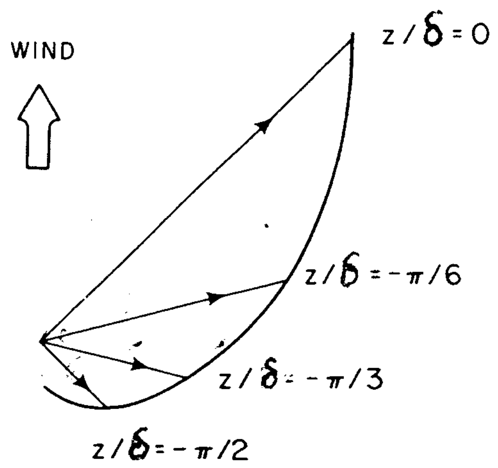

Figure 7.4.1: Ekman Boundary layer.

2. Physical Remark #2 :
   The total flux in Ekman layer is

$$\begin{aligned} U_x^E + iV_y^E &= \int_{-\infty}^{0} dz\ (u_E + i\, v_E) = \int_{-\infty}^{0} dz\, q_E \\ &= \frac{\delta\, u_*^2}{2\nu}\,(1+i) \int_{-\infty}^{0} e^{(1+i)z/\delta} dz \\ &= \frac{\delta\, u_*^2}{2\nu}\,(1+i) \cdot \frac{\delta}{1+i} \cdot e^{(1+i)^{z/\delta}} \Big|_{-\infty}^{0} = \frac{\delta^2\, u_*^2}{2\nu} = \frac{u_*^2}{f}. \end{aligned}$$

   where

$$\delta^2 = \frac{2\nu}{f}.$$

   Therefore the total mass flux in Ekman layer is 90 degrees inclined with respect to wind.

3. Physical remark # 3:
   Note that the flux in the surface Ekman layer is counter balanced by the geostrophic return flow beneath. This implies the assumption that the bottom Ekman layer is very weak and contributes little to the flux. Let us check.

7.4.4 Bottom Ekman layer

The total flow is governed by

$$-fv = \nu \frac{\partial^2 u}{\partial z^2} \qquad fu = fu_g + \nu \frac{\partial^2 v}{\partial z^2}.$$

Let

$$u_E = u - u_g \qquad v_E = v \tag{7.4.23}$$

so that

$$-fv_E = \nu\,\frac{\partial^2 u_E}{\partial z^2}, \qquad fu_E = \nu\,\frac{\partial^2 v_E}{\partial z^2} \tag{7.4.24}$$

Let us shift to new coordinates with the origin on the sea bed so that the boundary conditions are

$$z \to \infty, \quad u_E, v_E \to 0 \tag{7.4.25}$$

and

$$z = 0, \quad u_E = -u_g, v_E = 0 \tag{7.4.26}$$

Let

$$q_E = u_E + iv_E \quad q_E = A\,e^{-(1+i)z/\delta} \tag{7.4.27}$$

Since there is no slip at $z = 0$

$$q_E = -u_g.$$

We conclude,

$$A = -u_g.$$

and

$$\begin{aligned} q_E &= (u_E + iv_E) = -u_g\,e^{-(1+i)z/\delta} \\ &= -u_G\,e^{-z/\delta}\left(\cos\frac{z}{\delta} - i\,\sin\frac{z}{\delta}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} u_E &= -u_g\,e^{-z/\delta}\,\cos\frac{z}{\delta} \\ v_E &= u_g\,e^{-z/\delta}\,\sin\frac{z}{\delta}. \end{aligned}$$

The bottom shear stress is

$$\nu\,\frac{\partial q_E}{\partial z} = \frac{1+i}{\delta}\,u_g\,e^{-(1+i)\,z/\delta}$$

at $z = 0$

$$\nu\,\left.\frac{\partial q_E}{\partial z}\right|_0 = \frac{1+i}{\delta}\,u_g = -\frac{1+i}{\delta}\,\frac{u_*^2}{fH}.$$

The total flux in bottom Ekman layer is

$$\begin{aligned} U_E + iV_E &= \int_0^\infty (u_E + iv_E)\,dz \\ &= -u_g\int_0^\infty e^{-(1+i/\delta)z}dz \\ &= -u_g\,\frac{1}{-(1+i)/\delta}\,\left.e^{-(1+i/\delta)z}\right|_0^\infty \\ &= -u_g\,\frac{\delta}{1+i} = \frac{u_*^2}{fH}\,\frac{\delta}{1+i} = \frac{\delta}{2}(1-i)\frac{u_*^2}{fH} \end{aligned}$$

It is of order δ and is directed at 135 degrees to the right of wind.

7.4.5 Summary

:

- Total Ekman layer flux on top is u_*^2/f
- Total geostropic flux is $-u_*^2/f$
- Total bottom Ekman layer flux is very small $O\left[\frac{u_*^2}{f}\left(\frac{\delta}{H}\right)\right]$.