12

Engineering Theory of Prismatic Members

12–1. INTRODUCTION

St. Venant's theory of flexure-torsion is restricted to the case where—

1. There are *no* surface forces applied to the cylindrical surface.
2. The end cross sections can warp freely.

The warping function ϕ consists of a term due to flexure (ϕ_f) and a term due to pure torsion (ϕ_t). Since ϕ is *independent* of x_1, the linear expansion

$$\sigma_{11} = \frac{F_1}{A} + \frac{M_2}{I_2} x_3 - \frac{M_3}{I_3} x_2 \tag{12–1}$$

is the *exact* solution† for σ_{11}. The total shearing stress is given by

$$\sigma_{1S} = \sigma_t + \sigma_f \tag{12–2}$$

where σ_t is the pure-torsion distribution (due to ϕ_t) and σ_f represents the flexural distribution (due to ϕ_f). We generally determine σ_f by applying the engineering theory of shear stress distribution, which assumes that the cross section is rigid with respect to in-plane deformation. Using (12–1) leads to the following expression for the flexural shear flow (see (11–106)):

$$q_B = q_A - \frac{Q_3}{I_3} F_2 - \frac{Q_2}{I_2} F_3 \tag{12–3}$$

The warping function will depend on x_1 if forces are applied to the cylindrical surface or the ends are restrained with respect to warping. A term due to *variable* warping must be added to the linear expansion for σ_{11}. This leads to an additional term in the expression for the flexural shear flow. Since (12–1)

† A linear variation of normal stress is exact for a homogeneous beam. Composite beams (e.g., a sandwich beam) are treated by assuming a linear variation in extensional strain and obtaining the distributions of σ_{11} from the stress-strain relation. See Probs. 11–14 and 12–1.

satisfies the definition equations for F_1, M_2, M_3 identically, the normal stress correction is self-equilibrating; i.e., it is statically equivalent to zero. Also, the shear flow correction is statically equivalent to only a torsional moment since (12–3) satisfies the definition equations for F_2, F_3 identically.

In the engineering theory of members, we *neglect* the effect of variable warping on the normal and shearing stress; i.e., we use the stress distribution predicted by the St. Venant theory, which is based on *constant* warping and *no* warping restraint at the ends. In what follows, we develop the governing equations for the engineering theory and illustrate the two general solution procedures. This formulation is restricted to the *linear geometric* case. In the next chapter, we present a more refined theory which accounts for warping restraint, and investigate the error involved in the engineering theory.

12–2. FORCE-EQUILIBRIUM EQUATIONS

In the engineering theory, we take the stress resultants and couples referred to the centroid as force quantities, and determine the stresses using (12–1), (12–3), and the pure-torsional distribution due to M_T. To establish the force-equilibrium equations, we consider the differential element shown in Fig. 12–1. The statically equivalent external force and moment vectors per unit

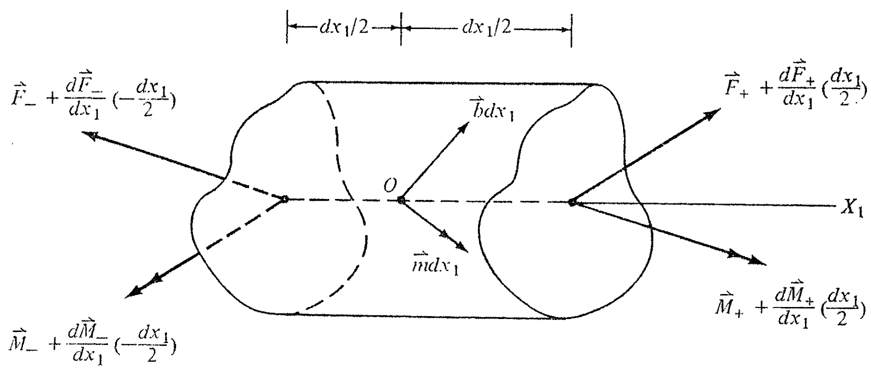


Fig. 12–1. Differential element for equilibrium analysis.

length along X_1 are denoted by $\vec{b}$, $\vec{m}$. Summing forces and moments about 0 leads to the following vector equilibrium equations (note that $\vec{F}_- = -\vec{F}_+$, $\vec{M}_- = -\vec{M}_+$):

$$\begin{aligned} \frac{d\vec{F}_+}{dx_1} + \vec{b} &= \vec{0} \\ \frac{d\vec{M}_+}{dx_1} + \vec{m} + (\vec{i}_1 \times \vec{F}_+) &= \vec{0} \end{aligned} \tag{a}$$

We obtain the scalar equilibrium equations by introducing the component expansions and equating the coefficients of the unit vectors to zero. The resulting system uncouples into four sets of equations that are associated with stretching, flexure in the X_1-X_2 plane, flexure in the X_1-X_3 plane, and twist.

Stretching

$$\frac{dF_1}{dx_1} + b_1 = 0$$

Flexure in X_1-X_2 Plane

$$\frac{dF_2}{dx_1} + b_2 = 0$$

$$\frac{dM_3}{dx_1} + m_3 + F_2 = 0 \qquad (12\text{–}4)$$

Flexure in X_1-X_3 Plane

$$\frac{dF_3}{dx_1} + b_3 = 0$$

$$\frac{dM_2}{dx_1} + m_2 - F_3 = 0$$

Twist

$$\frac{dM_1}{dx_1} + m_1 = 0$$

This uncoupling is characteristic only of *prismatic* members; the equilibrium equations for an arbitrary curved member are generally coupled, as we shall show in Chapter 15.

The flexure equilibrium equations can be reduced by solving for the shear force in terms of the bending moment, and then substituting in the remaining equations. We list the results below for future reference.

Flexure in X_1-X_2 Plane

$$F_2 = -\frac{dM_3}{dx_1} - m_3$$

$$\frac{d^2M_3}{dx_1^2} + \frac{dm_3}{dx_1} - b_2 = 0$$

$$(12\text{–}5)$$

Flexure in X_1-X_3 Plane

$$F_3 = \frac{dM_2}{dx_1} + m_2$$

$$\frac{d^2M_2}{dx_1^2} + \frac{dm_2}{dx_1} + b_3 = 0$$

Note that the shearing force is known once the bending moment variation is determined.

The statically equivalent external force and moment components acting on the end cross sections are called *end forces*. We generally use a bar superscript to indicate an end action in this text. Also, we use A, B to denote the negative and positive end points (see Fig. 12–2) and take the positive sense of an end

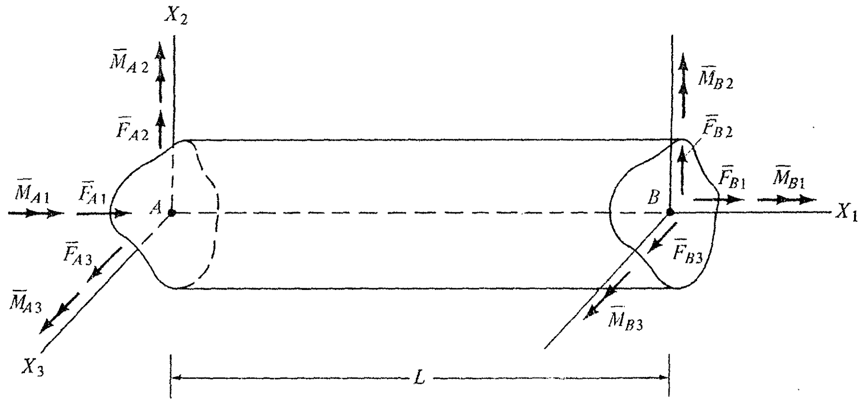


Fig. 12–2. Notation and positive direction for end forces.

force to *coincide* with the corresponding coordinate axis. The end forces are related to the stress resultants and couples by

$$\begin{aligned} \bar{F}_{Bj} &= [F_j]_{x_1=L} \\ \bar{M}_{Bj} &= [M_j]_{x_1=L} \\ \bar{F}_{Aj} &= -[F_j]_{x_1=0} \\ \bar{M}_{Aj} &= -[M_j]_{x_1=0} \end{aligned} \qquad (j = 1, 2, 3) \qquad (12\text{–}6)$$

A minus sign is required at A, since it is a negative face.

12–3. FORCE-DISPLACEMENT RELATIONS; PRINCIPLE OF VIRTUAL FORCES

We started by selecting the stress resultants and stress couples as force parameters. Applying the equilibrium conditions to a differential element results in a set of six differential equations relating the six force parameters. To complete the formulation, we must select a set of displacement parameters and relate the force and displacement parameters. These equations are generally called force-displacement relations. Since we have six equilibrium equations, we must introduce six displacement parameters in order for the formulation to be consistent.

Now, the force parameters are actually the statically equivalent forces and moments acting at the *centroid*. This suggests that we take as displacement parameters the equivalent rigid body translations and rotations of the cross section at the centroid. We define $\bar{u}$ and $\bar{\omega}$ as

$$\begin{aligned} \bar{u} &= \sum u_j \bar{\imath}_j = \text{equivalent rigid body translation vector at the centroid} \\ \bar{\omega} &= \sum \omega_j \bar{\imath}_j = \text{equivalent rigid body rotation vector} \end{aligned} \qquad (12\text{–}7)$$

By equivalent displacements, we mean

$$\iint_A (\text{force intensity})\,(\text{displacement})\,dA = \bar{F}\cdot\bar{u} + \bar{M}\cdot\bar{\omega} \qquad (12\text{–}8)$$

Note that (12–7) corresponds to a linear distribution of displacements over the cross section, whereas the actual distribution is nonlinear, owing to shear deformation. In this approach, we are allowing for an *average* shear deformation determined such that the energy is invariant.

We establish the force-displacement relations by applying the principle of virtual forces to the differential element shown in Fig. 12–3. The virtual-force

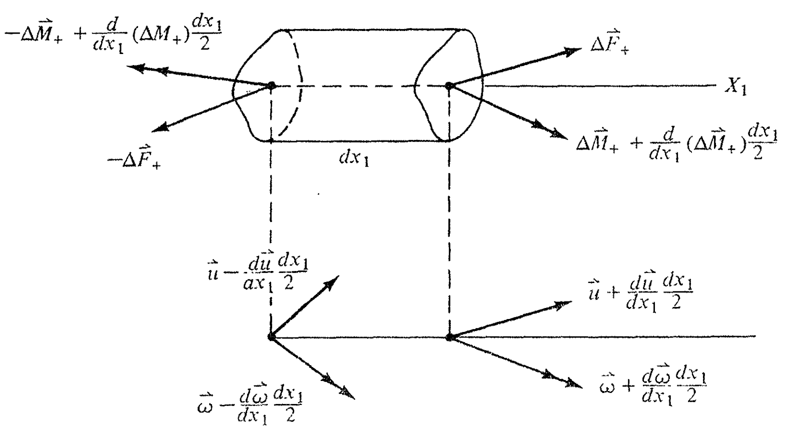


Fig. 12–3. Statically permissible force system.

system is statically permissible; that is, it satisfies the one-dimensional equilibrium equations

$$\begin{aligned} &\frac{d}{dx_1}(\Delta\bar{F}_+) = \bar{0} \\ &\frac{d}{dx_1}(\Delta\bar{M}_+) + (\bar{\imath}_1 \times \Delta\bar{F}_+) = \bar{0} \end{aligned} \qquad (a)$$

Specializing the principle of virtual forces for the one-dimensional elastic case, we can write

$$d\bar{V}^*\,dx_1 = \sum d_i\,\Delta P_i \qquad (b)$$

where d_i represents a displacement quantity, and P_i is the external force quantity corresponding to d_i. The term $d\bar{V}^*$ is the first-order change in the one-dimensional complementary energy density due to increments in the stress resultants and couples.

Evaluating the right-hand side of (b), we have

$$\sum d_i\,\Delta P_i = \left[\Delta\bar{F}_+\cdot\frac{d\bar{u}}{dx_1} + \Delta\bar{M}_+\cdot\frac{d\bar{\omega}}{dx_1} + \left(\frac{d}{dx_1}\Delta\bar{M}_+\right)\cdot\bar{\omega}\right]dx_1 \qquad (c)$$

Using the second equation in (a), (c) takes the form

$$\sum d_i\,\Delta P_i = \left[\Delta\bar{F}_+\cdot\left(\frac{d\bar{u}}{dx_1} + \bar{\imath}_1 \times \bar{\omega}\right) + \Delta\bar{M}_+\cdot\frac{d\bar{\omega}}{dx_1}\right]dx_1 \qquad (d)$$

Finally, evaluating the products, we obtain

$$\begin{aligned} \sum d_i\,\Delta P_i = [&\Delta F_1 u_{1,1} + \Delta F_2(u_{2,1} - \omega_3) + \Delta F_3(u_{3,1} + \omega_2) \\ &+ \Delta M_1\omega_{1,1} + \Delta M_2\omega_{2,1} + \Delta M_3\omega_{3,1}]dx_1 \end{aligned} \qquad (12\text{–}9)$$

Continuing, we expand $d\bar{V}^*$:

$$\begin{aligned} d\bar{V}^* &= \sum_{j=1}^{3}\left(\frac{\partial\bar{V}^*}{\partial F_j}\Delta F_j + \frac{\partial\bar{V}^*}{\partial M_j}\Delta M_j\right) \\ &= \sum_{j=1}^{3}(e_j\,\Delta F_j + k_j\,\Delta M_j) \end{aligned} \qquad (12\text{–}10)$$

The quantities e_j and k_j are *one-dimensional deformation measures.* Equating (12–9) and (12–10) leads to the following relation between the deformation measures and the displacements:

$$\begin{aligned} e_1 &= \frac{\partial\bar{V}^*}{\partial F_1} = u_{1,1} & k_1 &= \frac{\partial\bar{V}^*}{\partial M_1} = \omega_{1,1} \\ e_2 &= \frac{\partial\bar{V}^*}{\partial F_2} = u_{2,1} - \omega_3 & k_2 &= \frac{\partial\bar{V}^*}{\partial M_2} = \omega_{2,1} \\ e_3 &= \frac{\partial\bar{V}^*}{\partial F_3} = u_{3,1} + \omega_2 & k_3 &= \frac{\partial\bar{V}^*}{\partial M_3} = \omega_{3,1} \end{aligned} \qquad (12\text{–}11)$$

We see that—

1. e_1 is the average extensional strain.
2. e_2, e_3 are average transverse shear deformations.
3. k_1 is a twist deformation.
4. k_2, k_3 are average bending deformation measures (relative rotations of the cross section about X_2, X_3).

Once the form of $\bar{V}^*$ is specified, we can evaluate the partial derivatives. In what follows, we suppose that the material is linearly elastic. We allow for the possibility of an initial extensional strain, but no initial shear strain. The general expression for $\bar{V}^*$ is

$$\bar{V}^* = \iint_A \left[\frac{1}{2E}\sigma_{11}^2 + \sigma_{11}\varepsilon_1^0 + \frac{1}{2G}(\sigma_{12}^2 + \sigma_{13}^2)\right]dA \qquad (a)$$

where ε_1^0 denotes the initial extensional strain. Now, $\bar{V}^*$ for unrestrained torsion-flexure is given by (11–98). Since we are using the engineering theory of shear stress distribution, it is inconsistent to retain terms involving in-plane deformation, i.e., v_1/E. Adding terms due to $\sigma_1 = F_1/A$, $\sigma_1\varepsilon_1^0$, and *neglecting* the coupling between F_2, F_3 leads to

$$\bar{V}^* = F_1 e_1^0 + \frac{1}{2AE}F_1^2 + \frac{1}{2GA_2}F_2^2 + \frac{1}{2GA_3}F_3^2 + \frac{1}{2GJ}M_T^2 + k_2^0 M_2 + \frac{1}{2EI_2}M_2^2 + k_3^0 M_3 + \frac{1}{2EI_3}M_3^2 \tag{12-12}$$

where

$$M_T = M_1 + F_2\bar{x}_3 - F_3\bar{x}_2$$
$$e_1^0 = \frac{1}{A}\iint \varepsilon_1^0\, dA$$
$$k_2^0 = \frac{1}{I_2}\iint x_3\varepsilon_1^0\, dA$$
$$k_3^0 = \frac{-1}{I_3}\iint x_2\varepsilon_1^0\, dA$$

We take (12–12) as the definition of the one-dimensional linearly elastic complementary energy density for the *engineering* theory. One can interpret e_1^0, k_2^0, k_3^0 as "weighted" or equivalent initial strain measures.

Differentiating (12–12) with respect to the stress resultants and couples, and substituting in (12–11), we obtain the following force-displacement relations:

$$e_1 = e_1^0 + \frac{F_1}{AE} = u_{1,1} \qquad k_1 = \frac{M_T}{GJ} = \omega_{1,1}$$
$$e_2 = \frac{F_2}{GA_2} + \frac{M_T}{GJ}\bar{x}_3 = u_{2,1} - \omega_3 \qquad k_2 = k_2^0 + \frac{M_2}{EI_2} = \omega_{2,1} \tag{12-13}$$
$$e_3 = \frac{F_3}{GA_3} - \frac{M_T}{GJ}\bar{x}_2 = u_{3,1} + \omega_2 \qquad k_3 = k_3^0 + \frac{M_3}{EI_3} = \omega_{3,1}$$

To interpret the coupling between the shear and twist deformations, we note (see Fig. 12–4) that

$$u_2 = \bar{x}_3\omega_1$$
$$u_3 = -\bar{x}_2\omega_1 \tag{a}$$

defines the *centroidal* displacements due to a rigid body rotation about the *shear center*. Comparing (a) with (12–13), we see that the cross section twists about the *shear center*, not the centroid. This result is a consequence of neglecting the in-plane deformation terms in $\bar{V}^*$, i.e., of using (12–12).

Instead of working with centroidal quantities (M_1, u_2, u_3), we could have started with M_T and the translations of the *shear center*. This presupposes that the cross section rotates about the shear center. We replace u_2, u_3 (see Fig. 12–4) by

$$u_2 = u_{S2} + \omega_1\bar{x}_3$$
$$u_3 = u_{S3} - \omega_1\bar{x}_2 \tag{12-14}$$

where u_{S2}, u_{S3} denote the translations of the shear center. The terms involving F_2, F_3, M_1 in (12–9) transform to

$$\Delta M_T\omega_{1,1} + \Delta F_2(u_{S2,1} - \omega_3) + \Delta F_3(u_{S3,1} + \omega_2) \tag{a}$$

Then, taking M_T as an *independent* force parameter, we obtain

$$\frac{M_T}{GJ} = \omega_{1,1}$$
$$\frac{F_2}{GA_2} = u_{S2,1} - \omega_3 \tag{12-15}$$
$$\frac{F_3}{GA_2} = u_{S3,1} + \omega_2$$

Since the section twists about the shear center, it is more convenient to work with M_T and the translations of the shear center. Once u_{S2}, u_{S3}, and ω_1 are

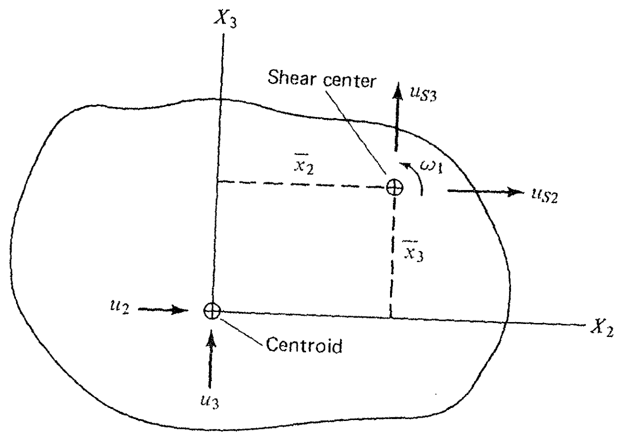


Fig. 12–4. Translations of the centroid and the shear center.

known, we can determine u_2, u_3 from (12–14). We list the uncoupled sets of force-displacement relations below for future reference.

Stretching

$$e_1^0 + \frac{F_1}{AE} = u_{1,1}$$

Flexure in X_1-X_2 Plane

$$\frac{F_2}{GA_2} = u_{S2,1} - \omega_3$$
$$k_3^0 + \frac{M_3}{EI_3} = \omega_{3,1}$$

Flexure in X_1-X_3 Plane (12–16)

$$\frac{F_3}{GA_3} = u_{S3,1} + \omega_2$$

$$k_2^0 + \frac{M_2}{EI_2} = \omega_{2,1}$$

Twist About the Shear Center

$$\frac{M_T}{GJ} = \omega_{1,1}$$

The development presented above is restricted to an elastic material. Now, the principle of virtual forces applies for an arbitrary material. Instead of first specializing it for the elastic case, we could have started with its general form (see (10–94)),

$$\int_{x_1} \left[\iint_A \varepsilon^T \Delta\sigma \, dA \right] dx_1 = \sum d_i \, \Delta P_i \tag{12–17}$$

where ε represents the *actual* strain matrix, and $\Delta\sigma$ denotes a system of statically permissible stresses due to the external force system, ΔP_i. We express the integral as

$$\iint_A \varepsilon^T \Delta\sigma \, dA = \sum_{j=1}^{3} (e_j \, \Delta F_j + k_j \, \Delta M_j) \tag{12–18}$$

and determine e_j, k_j, using $\Delta\sigma$ as defined by the engineering theory. For example, taking

$$\Delta\sigma_1 = \frac{\Delta F_1}{A} + \frac{\Delta M_2}{I_2} x_3 - \frac{\Delta M_3}{I_3} x_2 \tag{a}$$

leads to

$$e_1 = \frac{1}{A} \iint \varepsilon_1 \, dA$$

$$k_2 = \frac{1}{I_2} \iint x_3 \varepsilon_1 . dA \tag{b}$$

$$k_3 = \frac{-1}{I_3} \iint x_2 \varepsilon_1 \, dA$$

Once the extensional strain distribution is known, we can evaluate (b).

Using (12–18), the one-dimensional principle of virtual forces takes the form

$$\int_{x_1} [\sum (e_j \, \Delta F_j + k_j \, \Delta M_j)] dx_1 = \sum d_i \, \Delta P_i \tag{12–19}$$

The virtual-force system must satisfy the one-dimensional equilibrium equations (12–4). One should note that (12–19) is applicable for an *arbitrary* material. When the material is elastic, the bracketed term is equal to $d\bar{V}^*$, and we can write it as

$$\int_{x_1} d\bar{V}^* \, dx_1 = \sum d_i \, \Delta P_i \tag{12–20}$$

The expanded form for the linearly elastic case is

$$\begin{aligned} \int_{x_1} \Bigg[\left(e_1^0 + \frac{F_1}{AE} \right) \Delta F_1 + \left(\frac{F_2}{GA_2} \right) \Delta F_2 + \left(\frac{F_3}{GA_3} \right) \Delta F_3 + \frac{M_T}{GJ} \Delta M_T \\ + \left(k_2^0 + \frac{M_2}{EI_2} \right) \Delta M_2 + \left(k_3^0 + \frac{M_3}{EI_3} \right) \Delta M_3 \Bigg] dx_1 = \sum d_i \, \Delta P_i \end{aligned} \tag{12–21}$$

We use (12–21) in the force method discussed in Sec. 12–6.

12–4. SUMMARY OF THE GOVERNING EQUATIONS

At this point, we summarize the governing equations for the linear engineering theory of prismatic members. We list the equations according to the different modes of deformation (stretching, flexure, etc.). The boundary conditions reduce to either a force or the corresponding displacement is prescribed at each end.

Stretching (F_1, u_1)

$$\begin{aligned} F_{1,1} + b_1 &= 0 \\ e_1^0 + \frac{F_1}{AE} &= u_{1,1} \\ F_1 \text{ or } u_1 \text{ prescribed at } x_1 &= 0, L \end{aligned} \tag{12–22}$$

Flexure in X_1-X_2 Plane $(F_2, M_3, u_2, \omega_3)$

$$\begin{aligned} &F_{2,1} + b_2 = 0 \\ &M_{3,1} + m_3 + F_2 = 0 \\ &\frac{F_2}{GA_2} = u_{2,1} - \omega_3 \\ &\frac{M_3}{EI_3} + k_3^0 = \omega_{3,1} \\ &u_2 \text{ or } F_2 \text{ prescribed at } x_1 = 0, L \\ &M_3 \text{ or } \omega_3 \text{ prescribed at } x_1 = 0, L \end{aligned} \tag{12–23}$$

Flexure in the X_1-X_3 Plane $(F_3, M_2, u_3, \omega_2)$

$$\begin{aligned} &F_{3,1} + b_3 = 0 \\ &M_{2,1} + m_2 - F_3 = 0 \\ &\frac{F_3}{GA} = u_{3,1} + \omega_2 \\ &\frac{M_2}{EI_2} + k_2^0 = \omega_{2,1} \\ &u_3 \text{ or } F_3 \text{ prescribed at } x_1 = 0, L \\ &\omega_2 \text{ or } M_2 \text{ prescribed at } x_1 = 0, L \end{aligned} \tag{12–24}$$

Twist About the Shear Center $(M_T, \omega_1, u_2, u_3)$

$$M_{T,1} + m_T = 0$$
$$\frac{M_T}{GJ} = \omega_{1,1}$$
$$M_T \text{ or } \omega_1 \text{ prescribed at } x_1 = 0, L \qquad (12\text{–}25)$$
$$m_T = m_1 + b_2\bar{x}_3 - b_3\bar{x}_2$$
$$u_2 = \bar{x}_3\omega_1$$
$$u_3 = -\bar{x}_2\omega_1$$

12–5. DISPLACEMENT METHOD OF SOLUTION—PRISMATIC MEMBER

The displacement method involves integrating the governing differential equations and leads to expressions for the force and displacement parameters as functions of x_1. When the applied external loads are independent of the displacements, we can integrate the force-equilibrium equations directly and then find the displacements from the force-displacement relations. If the applied load depends on the displacements (e.g., a beam on an elastic foundation), we must first express the equilibrium equations in terms of the displacement parameters. This problem is more difficult, since it requires solving a differential equation rather than just successive integration. The following examples illustrate the application of the displacement method to a prismatic member.

Example 12–1

We consider the case where b_2 = const (Fig. E12–1). This loading will produce flexure in the X_1-X_2 plane and also twist about the shear center if the shear center does not lie on the X_2 axis. We solve the two uncoupled problems, superimpose the results, and then apply the boundary conditions.

Flexure in X_1-X_2 Plane

We start with the force-equilibrium equations,

$$F_{2,1} = -b_2 \qquad \text{(a)}$$
$$M_{3,1} = -F_2 \qquad \text{(b)}$$

Integrating (a), and noting that b_2 = const, we have

$$F_2 = F_2|_{x_1=0} - b_2x_1 \qquad \text{(c)}$$

For convenience, we use subscripts A, B for quantities associated with $x_1 = 0, L$:

$$F_j|_{x_1=0} = F_{Aj} \qquad F_j|_{x_1=L} = F_{Bj} \qquad \text{etc.} \qquad \text{(d)}$$

With this notation, (c) simplifies to

$$F_2 = F_{A2} - b_2x_1 \qquad \text{(e)}$$

Substituting for F_2 in (b), and integrating, we obtain

$$M_3 = M_{A3} - x_1F_{A2} + \tfrac{1}{2}b_2x_1^2 \qquad \text{(f)}$$

We consider next the force-displacement relations,

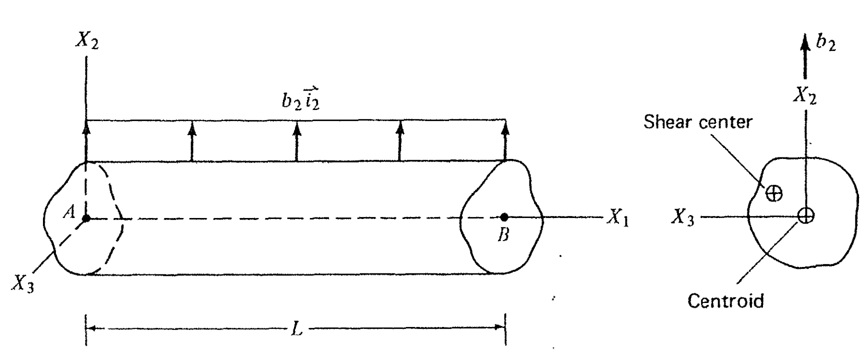
$$\omega_{3,1} = \frac{M_3}{EI_3} \qquad \text{(g)}$$
$$u_{2,1} = \omega_3 + \frac{F_2}{GA_2} \qquad \text{(h)}$$

Integrating (g) and then (h), we obtain

$$\omega_3 = \omega_{A3} + \frac{1}{EI_3}(x_1M_{A3} - \tfrac{1}{2}x_1^2F_{A2} + \tfrac{1}{6}b_2x_1^3)$$
$$u_2 = u_{A2} + x_1\omega_{A3} + F_{A2}\left(\frac{x_1}{GA_2} - \frac{x_1^3}{6EI_3}\right) + M_{A3}\left(\frac{x_1^2}{2EI_3}\right) + \frac{b_2x_1^2}{2}\left(-\frac{1}{GA_2} + \frac{x_1^2}{12EI_3}\right) \qquad \text{(i)}$$

The general flexual solution (for b_2 = const) is given by (e), (f), and (i).

Fig. E12–1



Twist About the Shear Center

The applied torsional moment with respect to the shear center is

$$m_T = b_2\bar{x}_3 \qquad \text{(j)}$$

Substituting for m_T in the governing equations,

$$M_{T,1} = -m_T$$
$$\omega_{1,1} = \frac{M_T}{GJ} \qquad \text{(k)}$$

and integrating, we obtain

$$M_T = M_{AT} - b_2\bar{x}_3x_1$$
$$\omega_1 = \omega_{A1} + \frac{1}{GJ}(x_1M_{AT} - \tfrac{1}{2}b_2\bar{x}_3x_1^2) \qquad \text{(l)}$$

The *additional* centroidal displacements due to twist are

$$u_2 = \bar{x}_3\omega_1$$
$$u_3 = -\bar{x}_2\omega_1 \qquad \text{(m)}$$

Cantilever Case

We suppose that the left end is fixed, and the right end is free. The boundary conditions are

$$\begin{aligned} u_{A2} &= \omega_{A3} = \omega_{A1} = 0 \\ F_{B2} &= M_{B3} = M_{BT} = 0 \end{aligned} \tag{n}$$

Specializing the general solution for these boundary conditions requires

$$\begin{aligned} F_{A2} &= b_2 L \\ M_{A3} &= \tfrac{1}{2} b_2 L^2 \\ M_{AT} &= b_2 \bar{x}_3 L \end{aligned} \tag{o}$$

and the final expressions reduce to

$$\begin{aligned} F_2 &= b_2(L - x_1) \\ M_3 &= b_2\left(\frac{L^2}{2} - Lx_1 + \frac{1}{2}x_1^2\right) \\ M_T &= b_2\bar{x}_3(L - x_1) \\ u_2 &= \bar{x}_3\omega_1 + b_2Lx_1\left(\frac{1}{GA_2} - \frac{1}{6}\frac{x_1^2}{EI_3}\right) + \frac{1}{2}b_2L^2\left(\frac{x_1^2}{2EI_3}\right) + \frac{b_2x_1^2}{2}\left(\frac{x_1^2}{12EI_3} - \frac{1}{GA_2}\right) \\ u_3 &= -\bar{x}_2\omega_1 \\ \omega_3 &= \frac{b_2}{EI_3}\left(\frac{x_1L^2}{2} - \frac{x_1^2L}{2} + \frac{x_1^3}{3}\right) \\ \omega_1 &= \frac{b_2\bar{x}_3x_1}{GJ}\left(L - \tfrac{1}{2}x_1\right) \end{aligned} \tag{p}$$

It is of interest to compare the deflections due to bending and shear deformation. Evaluating u_2 at $x_1 = L$, we have

$$\begin{aligned} u_{B2}\big|_{\text{bending}} &= \frac{1}{8}\frac{b_2L^4}{EI_3} = \delta_B \\ u_{B2}\big|_{\text{shear deformation}} &= \frac{1}{2}\frac{b_2L^2}{GA_2} = \delta_S \\ \frac{\delta_S}{\delta_B} &= \frac{E}{G}\frac{I_3}{L^2A_2} \end{aligned} \tag{q}$$

As an illustration, we consider a rectangular cross section and isotropic material with $\nu = 0.3$ (d = depth):

$$\begin{aligned} \frac{E}{G} &= 2.6 \\ \frac{I_3}{A_2} &= \frac{6}{5}\frac{I_3}{A} = \frac{d^2}{10} \\ \frac{\delta_S}{\delta_B} &= 1.04\left(\frac{d}{L}\right)^2 \end{aligned} \tag{r}$$

By definition, d/L is small with respect to unity for a member element and, therefore, it is

reasonable to neglect transverse shear deformation with respect to bending deformation for the isotropic case.† Formally, one sets $1/A_2 = 0$.

Fixed-End Case

We consider next the case where both ends are fixed. The boundary conditions are

$$\begin{aligned} u_{A2} &= \omega_{A3} = \omega_{A1} = 0 \\ u_{B2} &= \omega_{B3} = \omega_{B1} = 0 \end{aligned} \tag{s}$$

Specializing (h), (i), and (k) for this case, we obtain

$$\begin{aligned} F_{A2} &= \frac{b_2L}{2} \\ M_{A3} &= \frac{b_2L^2}{12} \\ M_{AT} &= \frac{1}{2}b_2\bar{x}_3L \end{aligned} \tag{t}$$

The final expressions are

$$\begin{aligned} F_2 &= b_2\left(\frac{L}{2} - x_1\right) \\ M_3 &= b_2\left(\frac{L^2}{12} - \frac{Lx_1}{2} + \frac{x_1^2}{2}\right) \\ M_T &= b_2\bar{x}_3\left(\frac{L}{2} - x_1\right) \\ u_2 &= \omega_1\bar{x}_3 + \frac{b_2}{2GA_2}(Lx_1 - x_1^2) + \frac{b_2}{24EI_3}(L^2x_1^2 - 2Lx_1^3 + x_1^4) \\ u_3 &= -\bar{x}_2\omega_1 \\ \omega_3 &= \frac{b_2}{EI_3}\left(\frac{L^2}{12}x_1 - \frac{L}{4}x_1^2 + \frac{1}{6}x_1^3\right) \\ \omega_1 &= \frac{b_2\bar{x}_3}{2GJ}(Lx_1 - x_1^2) \end{aligned} \tag{u}$$

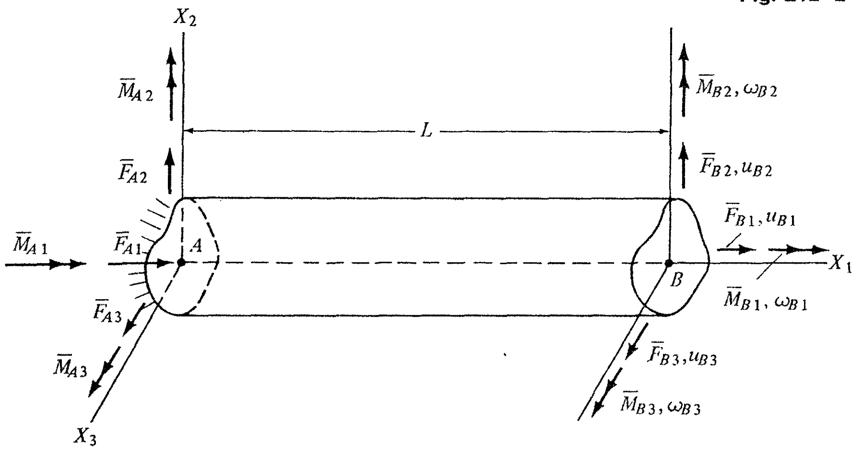
Example 12–2

We consider a member (Fig. E12–2) restrained at the left end, and subjected only to forces applied at the right end. We allow for the possibility of support movement at A. The expressions for the translations and rotations at B in terms of the end actions at B and support movement at A are called *member* force-displacement relations. We can obtain these relations for a prismatic member by direct integration of the force-displacement

† For shear deformation to be significant with respect to bending deformation, G/E must be of the same order as I/A_sL^2 where A_s is the *shear* area. This is not possible for the isotropic case. However, it may be satisfied for a sandwich beam having a *soft* core. See Prob. 12–1.

relations. In the next section, we illustrate an alternative approach, which utilizes the principle of virtual forces.†

Fig. E12–2



The boundary conditions at $x_1 = L$ are

$$[F_j]_{x_1=L} = \bar{F}_{Bj} \qquad [M_j]_{x_1=L} = \bar{M}_{Bj} \tag{a}$$

Integrating the force-equilibrium equations and applying (a) lead to the following expressions for the stress resultants and couples:

$$\begin{aligned} F_j &= \bar{F}_{Bj} \qquad (j = 1, 2, 3) \\ M_T &= \bar{M}_{BT} \\ M_2 &= \bar{M}_{B2} - (L - x_1)\bar{F}_{B3} \\ M_3 &= \bar{M}_{B3} + (L - x_1)\bar{F}_{B2} \end{aligned} \tag{b}$$

Using (b), the force-displacement relations take the form

$$\begin{aligned} u_{1,1} &= \frac{1}{AE}\bar{F}_{B1} \\ \omega_{3,1} &= \frac{1}{EI_3}[\bar{M}_{B3} + (L - x_1)\bar{F}_{B2}] \\ u_{2,1} &= \omega_3 + \frac{1}{GA_2}\bar{F}_{B2} + \frac{\bar{x}_3}{GJ}\bar{M}_{BT} \\ \omega_{2,1} &= \frac{1}{EI_2}[\bar{M}_{B2} - (L - x_1)\bar{F}_{B3}] \\ u_{3,1} &= -\omega_2 + \frac{1}{GA_3}\bar{F}_{B3} - \frac{\bar{x}_2}{GJ}\bar{M}_{BT} \\ \omega_{1,1} &= \frac{1}{GJ}\bar{M}_{BT} \end{aligned} \tag{c}$$

† See Prob. 12–11.

Integrating (c) and setting $x_1 = L$, we obtain

$$\begin{aligned} u_{B1} &= u_{A1} + \frac{L}{AE}\bar{F}_{B1} \\ \omega_{B3} &= \omega_{A3} + \frac{L}{EI_3}\bar{M}_{B3} + \frac{L^2}{2EI_3}\bar{F}_{B2} \\ u_{B2} &= u_{A2} + L\omega_{A3} + \frac{L^2}{2EI_3}\bar{M}_{B3} + \frac{L\bar{x}_3}{GJ}\bar{M}_{BT} + \left(\frac{L}{GA_2} + \frac{L^3}{3EI_3}\right)\bar{F}_{B2} \\ \omega_{B2} &= \omega_{A2} + \frac{L}{EI_2}\bar{M}_{B2} - \frac{L^2}{2EI_2}\bar{F}_{B3} \\ u_{B3} &= u_{A3} - L\omega_{A2} - \frac{L^2}{2EI_2}\bar{M}_{B2} - \frac{L\bar{x}_2}{GJ}\bar{M}_{BT} + \left(\frac{L}{GA_3} + \frac{L^3}{3EI_2}\right)\bar{F}_{B3} \\ \omega_{B1} &= \omega_{A1} + \frac{L}{GJ}\bar{M}_{BT} \end{aligned} \tag{d}$$

Finally, we replace $\bar{M}_{BT}$ by

$$\bar{M}_{BT} = \bar{M}_{B1} + \bar{x}_3\bar{F}_{B2} - \bar{x}_2\bar{F}_{B3} \tag{e}$$

and write the equations in matrix form:

$$\begin{Bmatrix} u_{B1} \\ u_{B2} \\ u_{B3} \\ \omega_{B1} \\ \omega_{B2} \\ \omega_{B3} \end{Bmatrix} = \begin{bmatrix} \frac{L}{AE} & & & & & \\ & \frac{L}{GA_2} + \frac{L^3}{3EI_3} + L\bar{x}_3^2/GJ & -\frac{L\bar{x}_3\bar{x}_2}{GJ} & \frac{L\bar{x}_3}{GJ} & & \frac{L^2}{2EI_3} \\ & -\frac{L\bar{x}_3\bar{x}_2}{GJ} & \frac{L}{GA_3} + \frac{L^3}{3EI_2} + L\bar{x}_2^2/GJ & -\frac{L\bar{x}_2}{GJ} & -\frac{L^2}{2EI_2} & \\ & \frac{\bar{x}_3 L}{GJ} & -\frac{\bar{x}_2 L}{GJ} & \frac{L}{GJ} & & \\ & & -\frac{L^2}{2EI_2} & & \frac{L}{EI_2} & \\ & \frac{L^2}{2EI_3} & & & & \frac{L}{EI_3} \end{bmatrix} \begin{Bmatrix} \bar{F}_{B1} \\ \bar{F}_{B2} \\ \bar{F}_{B3} \\ \bar{M}_{B1} \\ \bar{M}_{B2} \\ \bar{M}_{B3} \end{Bmatrix}$$
$$+ \{u_{A1}, u_{A2} + L\omega_{A3}, u_{A3} - L\omega_{A2}, \omega_{A1}, \omega_{A2}, \omega_{A3}\} \tag{f}$$

The coefficient matrix is called the member "flexibility" matrix and is generally denoted by $\mathbf{f}_B$.

We obtain expressions for the end forces in terms of the end displacements by inverting $\mathbf{f}$. The final relations are listed below for future reference:

$$
\begin{aligned}
\bar{F}_{B1} &= \frac{AE}{L}(u_{B1} - u_{A1}) \\
\bar{F}_{B2} &= \frac{12EI_3^*}{L^3}(u_{B2} - u_{A2}) - \frac{6EI_3^*}{L^2}(\omega_{B3} + \omega_{A3}) - \frac{12EI_3^*\bar{x}_3}{L^3}(\omega_{B1} - \omega_{A1}) \\
\bar{F}_{B3} &= \frac{12EI_2^*}{L^3}(u_{B3} - u_{A3}) + \frac{6EI_2^*}{L^2}(\omega_{B2} + \omega_{A2}) + \frac{12EI_2^*\bar{x}_2}{L^3}(\omega_{B1} - \omega_{A1}) \\
\bar{M}_{B1} &= \left[\frac{GJ}{L} + \frac{12E}{L^3}(\bar{x}_3^2 I_3^* + \bar{x}_2^2 I_2^*)\right](\omega_{B1} - \omega_{A1}) \\
&\quad - \frac{12EI_3^*\bar{x}_3}{L^3}(u_{B2} - u_{A2}) + \frac{6EI_3^*\bar{x}_3}{L^2}(\omega_{B3} + \omega_{A3}) \\
&\quad + \frac{12EI_2^*\bar{x}_2}{L^3}(u_{B3} - u_{A3}) + \frac{6EI_2^*\bar{x}_2}{L^2}(\omega_{B2} + \omega_{A2}) \\
\bar{M}_{B2} &= \frac{6EI_2^*}{L^2}(u_{B3} - u_{A3}) + \frac{6EI_2^*\bar{x}_2}{L^2}(\omega_{B1} - \omega_{A1}) \\
&\quad + (4 + a_2)\frac{EI_2^*}{L}\omega_{B2} + (2 - a_2)\frac{EI_2^*}{L}\omega_{A2} \\
\bar{M}_{B3} &= -\frac{6EI_3^*}{L^2}(u_{B2} - u_{A2}) + \frac{6EI_3^*\bar{x}_3}{L^2}(\omega_{B1} - \omega_{A1}) \\
&\quad + (4 + a_3)\frac{EI_3^*}{L}\omega_{B3} + (2 - a_3)\frac{EI_3^*}{L}\omega_{A3}
\end{aligned}
\tag{h}
$$

where

$$
a_2 = \frac{12EI_2}{GA_3L^2} \qquad a_3 = \frac{12EI_3}{GA_2L^2}
$$

$$
I_2^* = \frac{I_2}{1 + a_2} \qquad I_3^* = \frac{I_3}{1 + a_3}
$$

We introduce the assumption of negligible transverse shear deformation by setting $a_2 = a_3 = 0$.

The end forces at A and B are related by

$$
\begin{aligned}
\bar{F}_{Aj} &= -\bar{F}_{Bj} \qquad (j = 1, 2, 3) \\
\bar{M}_{A1} &= -\bar{M}_{B1} \\
\bar{M}_{A2} &= -\bar{M}_{B2} + L\bar{F}_{B3} \\
\bar{M}_{A3} &= -\bar{M}_{B3} - L\bar{F}_{B2}
\end{aligned}
\tag{i}
$$

We list only the expressions for $\bar{M}_{A2}$, $\bar{M}_{A3}$:

$$
\begin{aligned}
\bar{M}_{A2} &= \frac{6EI_2^*}{L^2}(u_{B3} - u_{A3}) + \frac{6EI_2^*\bar{x}_2}{L^2}(\omega_{B1} - \omega_{A1}) \\
&\quad + (4 + a_2)\frac{EI_2^*}{L}\omega_{A2} + (2 - a_2)\frac{EI_2^*}{L}\omega_{B2} \\
\bar{M}_{A3} &= -\frac{6EI_3^*}{L^2}(u_{B2} - u_{A2}) + \frac{6EI_3^*\bar{x}_3}{L^2}(\omega_{B1} - \omega_{A1}) \\
&\quad + (4 + a_3)\frac{EI_3^*}{L}\omega_{A3} + (2 - a_3)\frac{EI_3^*}{L}\omega_{B3}
\end{aligned}
\tag{j}
$$

Example 12-3

We consider next the case where the applied loads depend on the displacements. To simplify the discussion, we suppose the shear center is on the X_2 axis and the member is loaded only in the X_1-X_2 plane. The member will experience only flexure in the X_1-X_2 plane under these conditions.

The governing equations are given by (12-23):

$$
F_{2,1} + b_2 = 0 \tag{a}
$$

$$
F_2 = -M_{3,1} - m_3 \tag{b}
$$

$$
\omega_{3,1} = k_3^0 + \frac{M_3}{EI_3} \tag{c}
$$

$$
u_{2,1} - \omega_3 = \frac{F_2}{GA_2} \tag{d}
$$

An alternate form of (a) is

$$
M_{3,11} + m_{3,1} - b_2 = 0 \tag{e}
$$

Once M_3 is known, we can, using (b), find F_2.

Now, we solve (d) for ω_3 and substitute in (c):

$$
\begin{aligned}
\omega_3 &= u_{2,1} - \frac{F_2}{GA_2} \\
\omega_{3,1} &= u_{2,11} - \frac{F_{2,1}}{GA_2} = u_{2,11} + \frac{b_2}{GA_2}
\end{aligned}
\tag{f}
$$

Then,

$$
M_3 = EI_3\left(u_{2,11} + \frac{1}{GA_2}b_2 - k_3^0\right) \tag{g}
$$

and

$$
F_2 = -m_3 - EI_3\left(u_{2,111} + \frac{1}{GA_2}b_{2,1} - k_{3,1}^0\right) \tag{h}
$$

Finally, we substitute for M_3 in (e) and obtain a fourth-order differential equation involving u_2 and the load terms:

$$
\frac{d^4u_2}{dx_1^4} + \frac{d^2}{dx_1^2}\left(\frac{b_2}{GA_2} - k_3^0\right) + \frac{1}{EI_3}\left(\frac{dm_3}{dx_1} - b_2\right) = 0 \tag{i}
$$

The problem reduces to solving (i) and satisfying the boundary conditions:

$$
\left.\begin{array}{l} F_2 \text{ or } u_2 \text{ prescribed} \\ M_3 \text{ or } \omega_3 \text{ prescribed} \end{array}\right\} \text{at } x_1 = 0, L \tag{j}
$$

Neglecting transverse shear deformation simplifies the equations somewhat. The resulting equations are (we set $1/GA_2 = 0$)

$$
\begin{aligned}
\omega_3 &= u_{2,1} \\
M_3 &= EI_3(u_{2,11} - k_3^0) \\
F_2 &= -m_3 - EI_3(u_{2,111} - k_{3,1}^0) \\
\frac{d^4u_2}{dx_1^4} &- \frac{d^2}{dx_1^2}k_3^0 + \frac{1}{EI_3}\left(\frac{dm_3}{dx_1} - b_2\right) = 0
\end{aligned}
\tag{k}
$$

As an illustration, consider the case of linear restraint against translation of the centroid, e.g., a beam on a linearly elastic foundation. The distributed loading consists of two terms, one due to the applied external loading and the other due to the restraint force. We write

$$b_2 = q - ku_2 \tag{l}$$

where q denotes the external distributed load and k is the stiffness factor for the restraint. We suppose $m_3 = k_3^0 = 0$, k is constant, and transverse shear deformation is negligible. Specializing (k) for this case, we have

$$\begin{aligned} \omega_3 &= u_{2,1} \\ M_3 &= EI_3 u_{2,11} \\ F_2 &= -EI_3 u_{2,111} \end{aligned} \tag{m}$$

$$\frac{d^4 u_2}{dx_1^4} + \frac{k}{EI_3} u_2 = \frac{q}{EI_3} \tag{n}$$

$$\left.\begin{array}{l} F_2 \text{ or } u_2 \text{ prescribed} \\ M_3 \text{ or } \omega_3 \text{ prescribed} \end{array}\right\} \text{at } x_1 = 0, L \tag{o}$$

The general solution of (n) is

$$\begin{aligned} u_2 &= u_{2,p} + e^{-\lambda x_1}(C_1 \sin \lambda x_1 + C_2 \cos \lambda x_1) + e^{\lambda x_1}(C_3 \sin \lambda x_1 + C_4 \cos \lambda x_1) \\ \lambda &= \left(\frac{k}{4EI_3}\right)^{1/4} \end{aligned} \tag{p}$$

where $u_{2,p}$ represents the particular solution due to q. Enforcement of the boundary conditions at $x = 0$, L leads to the equations relating the four integration constants.

The function $e^{-\lambda x}$ decays with increasing x, whereas $e^{\lambda x}$ increases with increasing x. For $\lambda x > \approx 3$, $e^{-\lambda x} \approx 0$. If the member length L is greater than $2(3/\lambda) = 2L_b$ (we interpret L_b as the width of the boundary layer), we can approximate the solution by the following:

$$\begin{aligned} 0 \leqslant x_1 < L_b&: \quad u_2 = u_{2,p} + e^{-\lambda x_1}(C_1 \sin \lambda x_1 + C_2 \cos \lambda x_1) \\ L_B < x_1 < L - L_b&: \quad u_2 = u_{2,p} \\ L - L_b < x_1 \leqslant L&: \quad u_2 = u_{2,p} + e^{\lambda x_1}(C_3 \sin \lambda x_1 + C_4 \cos \lambda x_1) \end{aligned} \tag{q}$$

The constants (C_1, C_2) are determined from the boundary conditions at $x_1 = 0$ and (C_3, C_4) from the conditions at $x_1 = L$. Note that C_3 and C_4 must be of order $e^{-\lambda L}$ since u_2 is finite at $x_1 = L$.

Application 1

The boundary conditions at $x_1 = 0$ (Fig. E12–3A) are

$$\begin{aligned} u_2 &= 0 \\ M_3 &= EI_3 u_{2,11} = 0 \end{aligned}$$

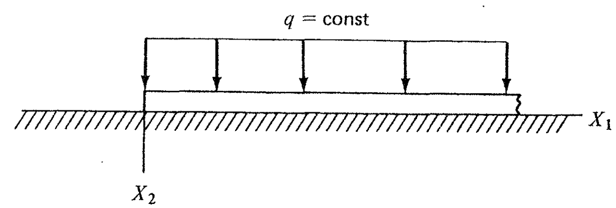
Since q is constant, the particular solution follows directly from (n),

$$u_{2,p} = q/k$$

The complete solution is

$$u_2 = \frac{q}{k}(1 - e^{-\lambda x_1} \cos \lambda x_1)$$

Fig. E12–3A



Application 2

The boundary conditions at $x_1 = 0$ (Fig. E12–3B) are

$$\begin{aligned} u_{2,1} &= 0 \\ F_2 &= -EI_3 u_{2,111} = -P/2 \end{aligned}$$

and the solution is

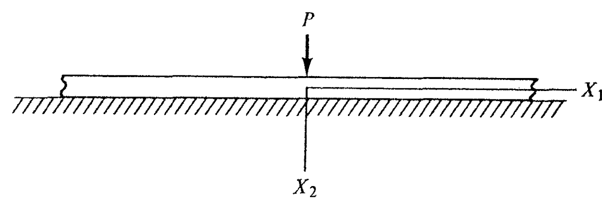
$$u_2 = \frac{P\lambda}{2k} e^{-\lambda x_1}(\cos \lambda x_1 + \sin \lambda x_1)$$

The four basic functions encountered are

$$\begin{aligned} \psi_1 &= e^{-\lambda x}(\cos \lambda x + \sin \lambda x) \\ \psi_2 &= e^{-\lambda x} \sin \lambda x = -\frac{1}{2\lambda}\psi_1' \\ \psi_3 &= e^{-\lambda x}(\cos \lambda x - \sin \lambda x) = \frac{1}{\lambda}\psi_2' \\ \psi_4 &= e^{-\lambda x} \cos \lambda x = -\frac{1}{2\lambda}\psi_3' \end{aligned} \tag{12–26}$$

Their values over the range from $\lambda x = 0$ to $\lambda x = 5$ are presented in Table 12–1.

Fig. E12–3B



12–6. FORCE METHOD OF SOLUTION

In the force method, we apply the principle of virtual forces to determine the displacement at a point and also to establish the equations relating the force redundants for a statically indeterminate member. We start with the one-dimensional form of the principle of virtual forces developed in Sec. 12–3 (see Equation 12–19):

$$\int_{x_1} [\textstyle\sum (e_j \, \Delta F_j + k_j \, \Delta M_j)] dx_1 = \sum d_i \, \Delta P_i \tag{a}$$

Table 12–1
Numerical Values of the ψ Functions

λx	ψ_1	ψ_2	ψ_3	ψ_4	λx
0.0	1.000	0.000	1.000	1.000	0
0.2	0.965	0.163	0.640	0.802	0.2
0.4	0.878	0.261	0.356	0.617	0.4
0.6	0.763	0.310	0.143	0.453	0.6
0.8	0.635	0.322	−0.009	0.313	0.8
1.0	0.508	0.310	−0.111	0.199	1.0
1.2	0.390	0.281	−0.172	0.109	1.2
1.4	0.285	0.243	−0.201	0.042	1.4
1.6	0.196	0.202	−0.208	−0.006	1.6
1.8	0.123	0.161	−0.199	−0.038	1.8
2.0	0.067	0.123	−0.179	−0.056	2.0
2.2	0.024	0.090	−0.155	−0.065	2.2
2.4	−0.006	0.061	−0.128	−0.067	2.4
2.6	−0.025	0.038	−0.102	−0.064	2.6
2.8	−0.037	0.020	−0.078	−0.057	2.8
3.0	−0.042	0.007	−0.056	−0.049	3.0
3.2	−0.043	−0.002	−0.038	−0.041	3.2
3.4	−0.041	−0.009	−0.024	−0.032	3.4
3.6	−0.037	−0.012	−0.012	−0.024	3.6
3.8	−0.031	−0.014	−0.004	−0.018	3.8
4.0	−0.026	−0.014	0.002	−0.012	4.0
4.2	−0.020	−0.013	0.006	−0.007	4.2
4.4	−0.016	−0.012	0.008	−0.004	4.4
4.6	−0.011	−0.010	0.009	−0.001	4.6
4.8	−0.008	−0.008	0.009	0.001	4.8
5.0	−0.005	−0.007	0.008	0.002	5.0

where e_j, k_j are the actual one-dimensional deformation measures;
d_i represents a displacement quantity;
ΔP_i is an external virtual force applied in the direction of d_i.

The relations between the deformation measures and the internal forces depend on the material properties and the assumed stress expansions. The appropriate relations for the linear elastic engineering theory are given by (12–13). If a displacement is prescribed, the corresponding force is actually a reaction. We use $\bar{d}_k$, ΔR_k to denote a prescribed displacement and the corresponding reaction increment, and write (a) as

$$\int_{x_1}[\sum(e_j\,\Delta F_j + k_j\,\Delta M_j)]dx_1 - \sum \bar{d}_k\,\Delta R_k = \sum d_i\,\Delta P_i \qquad (12\text{–}27)$$

where d_i represents an unknown displacement quantity.

To determine the displacement at some point, say Q, in the direction defined by the unit vector $\bar{t}_Q$, we apply a virtual force $\Delta P_Q \bar{t}_q$, and generate the necessary internal forces and reactions required for equilibrium using the one-dimensional force-equilibrium equations. We express the required virtual-force system as

$$\begin{aligned}\Delta F_j &= F_{j,Q}\,\Delta P_Q\\ \Delta M_j &= M_{j,Q}\,\Delta P_Q\\ \Delta R_k &= R_{k,Q}\,\Delta P_Q\end{aligned} \qquad (12\text{–}28)$$

Introducing (12–28) in (12–27) and canceling ΔP_Q leads to

$$d_Q = -\sum R_{k,Q}\,\bar{d}_k + \int_{x_1}[\sum(e_jF_{j,Q} + k_jM_{j,Q})]dx_1 \qquad (12\text{–}29)$$

This expression is applicable for an arbitrary material, but is restricted to the linear geometric case. Since the only requirement on the virtual force system is that it be statically permissible, one can always work with a statically *determinate* virtual force system. The expanded form of (12–29) for the linearly elastic case follows from (12–21):

$$\begin{aligned}d_Q = -\sum R_{k,Q}\,\bar{d}_k + \int_{x_1}\Bigg[&\left(e_1^0 + \frac{F_1}{AE}\right)F_{1,Q}\\ &+\left(\frac{F_2}{GA_2}\right)F_{2,Q} + \left(\frac{F_3}{GA_3}\right)F_{3,Q} + \frac{M_T}{GJ}M_{T,Q}\\ &+\left(k_2^0 + \frac{M_2}{EI_2}\right)M_{2,Q} + \left(k_3^0 + \frac{M_3}{EI_3}\right)M_{3,Q}\Bigg]dx_1\end{aligned} \qquad (12\text{–}30)$$

where

$$\begin{aligned}e_1^0 &= \frac{1}{A}\iint \varepsilon_1^0\,dA\\ k_2^0 &= \frac{1}{I_2}\iint x_3\varepsilon_1^0\,dA\\ k_3^0 &= \frac{-1}{I_3}\iint x_2\varepsilon_1^0\,dA\end{aligned}$$

Finally, we can express (12–29) for the elastic case in terms of V^*:

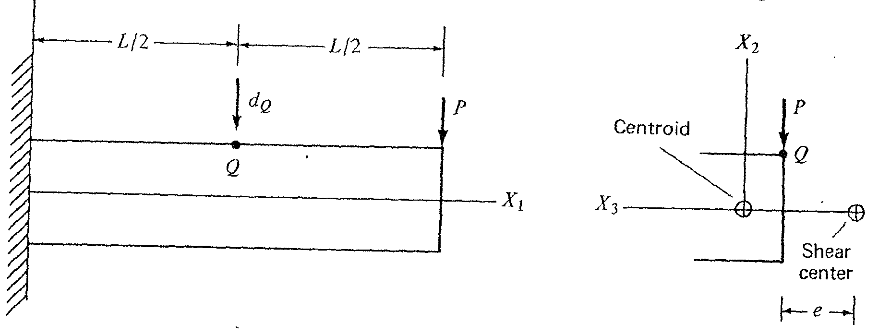
$$d_Q = \int_{x_1}\frac{\partial \bar{V}^*}{\partial P_Q}dx_1 - \sum \bar{d}_k\frac{\partial R_k}{\partial P_Q} \qquad (12\text{–}31)$$

This form follows from (12–20) and applies for an *arbitrary elastic* material.

Example 12–4

We consider the channel member shown in Fig. E12–4A. We suppose that the material is linearly elastic and that there is no support movement. We will determine the vertical

Fig. E12–4A



displacement of the web at point Q due to—

1. the concentrated force P
2. a temperature increase ΔT, given by

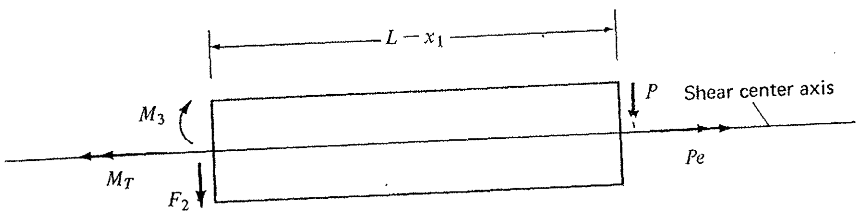
$$\Delta T = a_1 x_1 + a_2 x_1 x_2 + a_3 x_1 x_3$$

Force System Due to P

Applying the equilibrium conditions to the segment shown in Fig. E12–4B leads to

$$\begin{aligned} F_2 &= -P \\ M_T &= +Pe \\ M_3 &= -P(L - x_1) \\ F_1 &= F_3 = M_2 = 0 \end{aligned} \qquad \text{(a)}$$

Fig. E12–4B



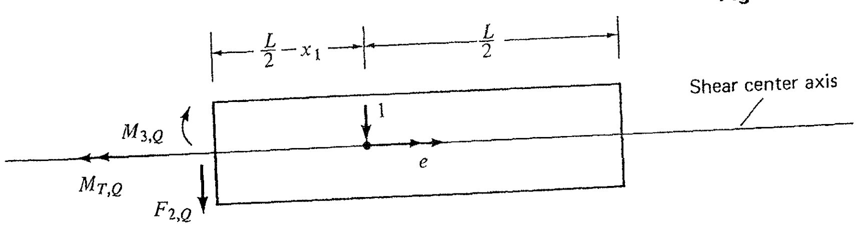
Virtual-Force System

We take d_Q positive when downward, i.e., in the $-X_2$ direction. To be consistent, we must apply a unit downward force at Q. The required internal forces follow from Fig. E12–4C:

$$0 \leq x_1 \leq \frac{L}{2} \qquad \left\{\begin{aligned} F_{2,Q} &= -1 \\ M_{T,Q} &= e \\ M_{3,Q} &= -\left(\frac{L}{2} - x_1\right) \\ F_{1,Q} &= F_{3,Q} = M_{2,Q} = 0 \end{aligned}\right\} \qquad \text{(b)}$$

$$\frac{L}{2} < x_1 \leq L \qquad \left\{\begin{aligned} F_{j,Q} &= 0 \\ M_{j,Q} &= 0 \end{aligned} \quad (j = 1, 2, 3)\right\} \qquad \text{(c)}$$

Fig. E12–4C



Initial Deformations

The initial extensional strain due to the temperature increase is

$$\varepsilon_1^0 = \alpha\,\Delta T = \alpha(a_1 x_1 + a_2 x_1 x_2 + a_3 x_1 x_3) \qquad \text{(d)}$$

The equivalent one-dimensional initial deformations are

$$\begin{aligned} e_1^0 &= \frac{1}{A}\iint \varepsilon_1^0\, dA = \alpha a_1 x_1 \\ k_2^0 &= \frac{1}{I_2}\iint x_3 \varepsilon_1^0\, dA = \alpha a_3 x_1 \\ k_3^0 &= \frac{-1}{I_3}\iint x_2 \varepsilon_1^0\, dA = -\alpha a_2 x_1 \end{aligned} \qquad \text{(e)}$$

Determination of d_Q

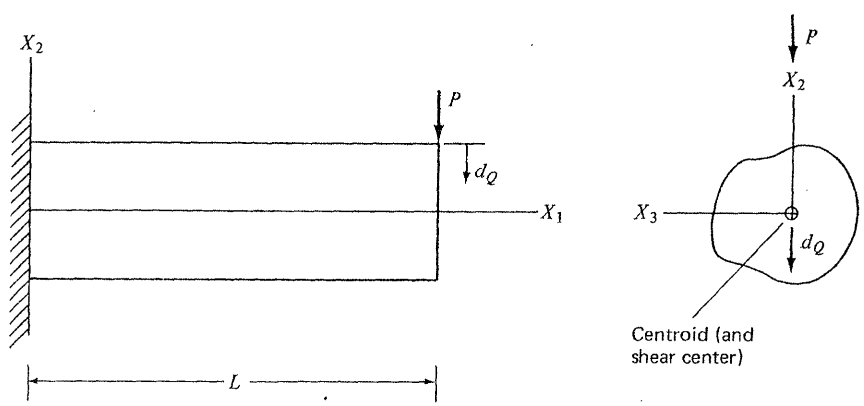
Substituting for the forces and initial deformations in (12–30), we obtain

$$\begin{aligned} d_Q &= \int_0^{L/2} \left\{\frac{P}{GA_2} + \frac{Pe^2}{GJ} + \left[\alpha a_2 x_1 + \frac{P}{EI_3}(L - x_1)\right]\left(\frac{L}{2} - x_1\right)\right\} dx_1 \\ &= P\left\{\frac{L}{2GA_2} + \frac{e^2 L}{2GJ} + \frac{5}{48}\frac{L^3}{EI_3}\right\} + \frac{\alpha a_2 L^2}{48} \end{aligned} \qquad \text{(f)}$$

Example 12–5

When the material is nonlinear, we must use (12–29) rather than (12–30). To illustrate the nonlinear case, we determine the vertical displacement due to P at the right end of the member shown in Fig. E12–5. We suppose that transverse shear deformation is negligible, and take the relation between k_3 and M_3 as

Fig. E12–5



$$k_3 = a_1 M_3 + a_3 M_3^3 \qquad \text{(a)}$$

Noting that only $F_{2,Q}$ and $M_{3,Q}$ are finite, and letting $e_2 = 0$, the general expression for d_Q reduces to

$$d_Q = \int_0^L k_3 M_{3,Q}\, dx_1 \tag{b}$$

Now,

$$\begin{aligned} M_3 &= -P(L - x_1) \\ M_{3,Q} &= -(L - x_1) \end{aligned} \tag{c}$$

Then,

$$k_3 = -Pa_1(L - x_1) - P^3 a_3 (L - x_1)^3 \tag{d}$$

Substituting for k_3 in (b), we obtain

$$d_Q = Pa_1 \frac{L^3}{3} + P^3 a_3 \frac{L^5}{5}$$

We describe next the application of the principle of virtual forces in the analysis of a statically indeterminate member. We suppose that the member is statically indeterminate to the rth degree. The first step involves selecting r force quantities, $Z_1, Z_2, \ldots, Z_r$. These quantities may be either internal forces or reactions, and are generally called *force redundants.*

Using the force-equilibrium equations, we express the internal forces and reactions in terms of the prescribed external forces and the force redundants.

$$\begin{aligned} F_j &= F_{j,0} + \sum_{k=1}^{r} F_{j,k} Z_k \\ M_j &= M_{j,0} + \sum_{k=1}^{r} M_{j,k} Z_k \\ R_i &= R_{i,0} + \sum_{k=1}^{r} R_{i,k} Z_k \end{aligned} \tag{12–32}$$

The member corresponding to $Z_1 = Z_2 = \cdots = Z_r = 0$ is conventionally called the primary structure. Note that all the force analyses are carried out on the *primary* structure. The set $(F_{j,0}, M_{j,0}, R_{i,0})$ represents the internal forces and reactions for the primary structure due to the prescribed external forces. Also, $(F_{j,k}, M_{j,k}, R_{i,k})$ represents the forces and reactions for the primary structure due to a unit value of Z_k. One must select the force resultants such that the resulting primary structure is *stable.*

Once the force redundants are known, we can find the total forces from (12–32). It remains to establish a system of r equations relating the force redundants. With this objective, we consider the virtual-force system consisting of ΔZ_k and the corresponding internal forces and reactions,

$$\begin{aligned} \Delta F_j &= F_{j,k}\, \Delta Z_k \\ \Delta M_j &= M_{j,k}\, \Delta Z_k \\ \Delta R_i &= R_{i,k}\, \Delta Z_k \end{aligned} \tag{a}$$

This system is statically permissible. Substituting (a) in (12–27), and noting that $\Delta P_i = 0$, we obtain

$$\int_{x_1} \left[\sum_j (e_j F_{j,k} + k_j M_{j,k}) \right] dx_1 = \sum_i \bar{d}_i R_{i,k} \tag{12–33}$$

Taking $k = 1, 2, \ldots, r$ results in a set of r equations relating the *actual* deformations. One can interpret these equations as *compatibility* conditions, since they represent restrictions on the deformations.

To proceed further, we must express the deformations in terms of F_j, M_j. In what follows, we suppose that the material is *linearly elastic.* The compatibility conditions for the linearly elastic case are given by

$$\begin{aligned} \int_{x_1} \Bigg[\left(e_1^0 + \frac{F_1}{AE}\right) F_{1,k} + \left(\frac{F_2}{GA_2}\right) F_{2,k} + \left(\frac{F_3}{GA_3}\right) F_{3,k} + \left(\frac{M_T}{GJ}\right) M_{T,k} \\ + \left(k_2^0 + \frac{M_2}{EI_2}\right) M_{2,k} + \left(k_3^0 + \frac{M_3}{EI_3}\right) M_{3,k} \Bigg] dx_1 = \sum \bar{d}_i R_{i,k} \\ k = 1, 2, \ldots, r \end{aligned} \tag{12–34}$$

A more compact form, which is valid for an arbitrary elastic material, is

$$\int_{x_1} \frac{\partial \bar{V}^*}{\partial Z_k}\, dx_1 = \sum \bar{d}_i \frac{\partial R_i}{\partial Z_k} \qquad (k = 1, 2, \ldots, r) \tag{12–35}$$

The final step involves substituting for F_j, M_j using (12–32). We write the resulting equations as

$$\sum_{j=1}^{r} f_{kj} Z_j = \Delta_k \qquad (k = 1, 2, \ldots, r) \tag{12–36}$$

where

$$\begin{aligned} f_{kj} = f_{jk} &= \int_{x_1} \Bigg[\frac{1}{AE} F_{1,j} F_{1,k} + \frac{1}{GA_2} F_{2,j} F_{2,k} + \frac{1}{GA_3} F_{3,j} F_{3,k} \\ &\quad + \frac{1}{GJ} M_{T,j} M_{T,k} + \frac{1}{EI_2} M_{2,j} M_{2,k} + \frac{1}{EI_3} M_{3,j} M_{3,k} \Bigg] dx_1 \\ \Delta_k &= \sum \bar{d}_i R_{i,k} - \int_{x_1} \Bigg[\left(e_1^0 + \frac{F_{1,0}}{AE}\right) F_{1,k} + \left(\frac{F_{2,0}}{GA_2}\right) F_{2,k} + \left(\frac{F_{3,0}}{GA_3}\right) F_{3,k} \\ &\quad + \left(\frac{M_{T,0}}{GJ}\right) M_{T,k} + \left(k_2^0 + \frac{M_{2,0}}{EI_2}\right) M_{2,k} + \left(k_3^0 + \frac{M_{3,0}}{EI_3}\right) M_{3,k} \Bigg] dx_1 \end{aligned}$$

The various terms in (12–36) have geometrical significance. Using (12–30), we see that f_{jk} is the displacement of the primary structure in the direction of Z_j due to a unit value of Z_k. Since $f_{jk} = f_{kj}$, it is also equal to the displacement in the direction of Z_k due to a unit value of Z_j. Generalizing this result, we can write

$$(d_i)_{P_j = 1} = (d_j)_{P_i = 1} \tag{12–37}$$

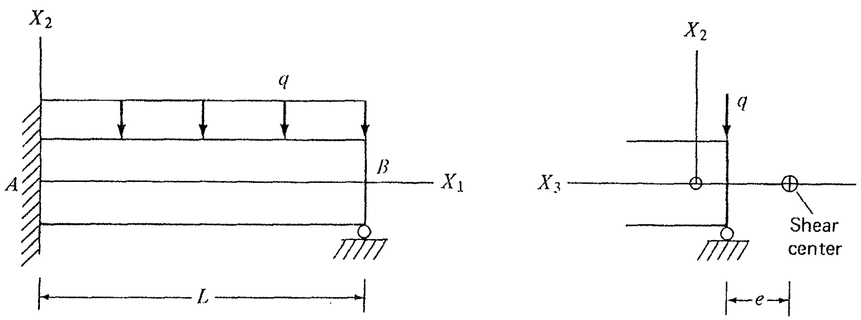
where i, j are arbitrary points, and P_n corresponds to d_n, i.e., i has the same direction and sense. Equation (12–37) is called *Maxwell's law of reciprocal deflections*, and follows directly from (12–30). The term Δ_k is the *actual* displacement of the point of application of Z_k, minus the displacement of the primary structure in the direction of Z_k due to support movement, initial strain, and the prescribed external forces. If we take Z_k as an internal force quantity (stress resultant or stress couple), Δ_k represents a relative displacement (translation or rotation) of adjacent cross sections.

One can interpret (12–36) as a superposition of the displacements due to the various effects. They are generally called *superposition* equations in elementary texts.† If the material is physically *nonlinear*, (12–36) are *not* applicable, and one must start with (12–33). The approach is basically the same as for the linear case. However, the final equations will be nonlinear. The following examples illustrate some of the details involved in applying the *force* method to statically indeterminate prismatic members.

Example 12–6

This loading (Fig. E12–6A) will produce flexure in the X_1-X_2 plane and twist about the shear center; i.e., only F_2, M_3 and M_T are finite. The member is indeterminate to the first degree. We will take the reaction at B as the force redundant.

Fig. E12–6A



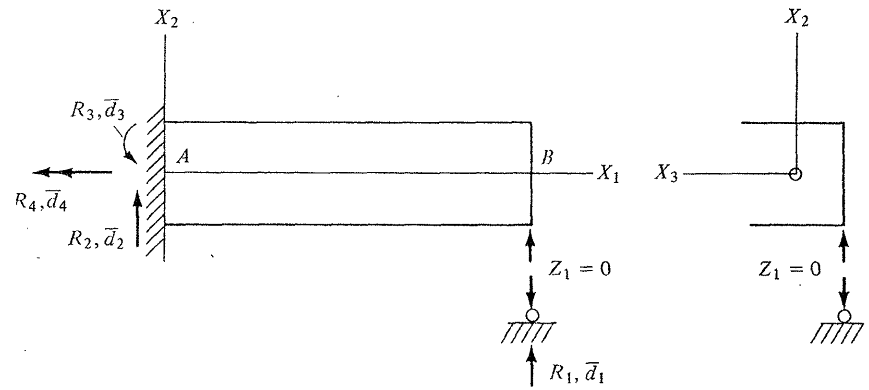
Primary Structure

One can select the positive sense of the reactions arbitrarily. (See Fig. E12–6B.) We work with the twisting moment with respect to the shear center. The reactions are related to the internal forces by

$$R_1 = Z_1$$
$$R_2 = -[F_2]_{X_1=0}$$
$$R_3 = -[M_3]_{X_1=0} \qquad \text{(a)}$$
$$R_4 = +[M_T]_{X_1=0}$$

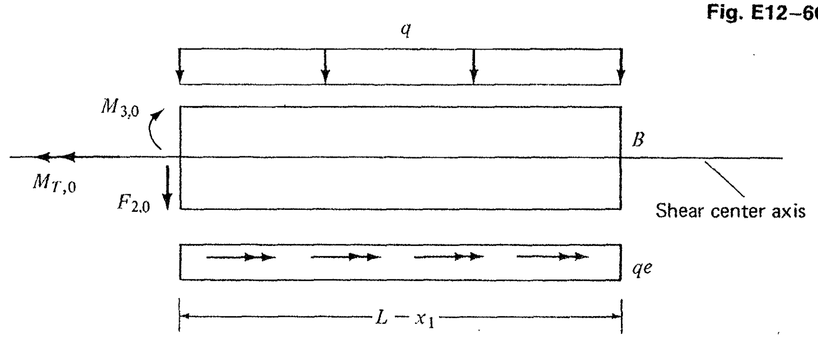
† See, for example, Art. 13–2 in Ref. 3.

Fig. E12–6B



Force System Due to Prescribed External Forces $(F_{j,0}, M_{j,0}, R_{i,0})$

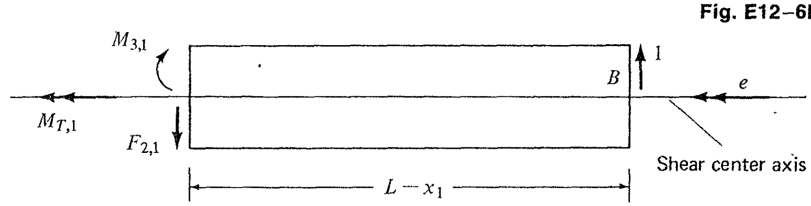
Fig. E12–6C



$$F_{2,0} = -q(L - x_1) \qquad R_{1,0} = 0$$
$$M_{T,0} = qe(L - x_1) \qquad R_{2,0} = qL$$
$$M_{3,0} = -\frac{q}{2}(L - x_1)^2 \qquad R_{3,0} = \frac{qL^2}{2} \qquad \text{(b)}$$
$$F_{1,0} = F_{3,0} = M_{2,0} = 0 \qquad R_{4,0} = qeL$$

Force System Due to $Z_1 = +1(F_{j,1}, M_{j,1}, R_{i,1})$

Fig. E12–6D



$$\begin{aligned} F_{2,1} &= +1 & R_{1,1} &= +1 \\ M_{T,1} &= -e & R_{2,1} &= -1 \\ M_{3,1} &= +(L - x_1) & R_{3,1} &= -L \\ F_{1,1} &= F_{3,1} = M_{2,1} = 0 & R_{4,1} &= -e \end{aligned} \qquad \text{(c)}$$

Equation for Z_1

We suppose that the member is linearly elastic. Specializing (12–36) for this problem,

$$\begin{aligned} f_{11}Z_1 &= \Delta_1 \\ f_{11} &= \int_0^L \left[\frac{1}{GA_2}(F_{2,1})^2 + \frac{1}{GJ}(M_{T,1})^2 + \frac{1}{EI_3}(M_{3,1})^2\right] dx_1 \\ \Delta_1 &= \sum_{i=1}^{4} \bar{d}_i R_{i,1} - \int_0^L \left[\frac{1}{GA_2}F_{2,0}F_{2,1} + \frac{1}{GJ}M_{T,0}M_{T,1} + \left(k_3^0 + \frac{M_{3,0}}{EI_3}\right) M_{3,1}\right] dx_1 \end{aligned} \qquad \text{(d)}$$

and then substituting for the forces and evaluating the resulting integrals, we obtain

$$\begin{aligned} f_{11} &= \frac{L}{GA_2} + \frac{Le^2}{GJ} + \frac{L^3}{3EI_3} \\ \Delta_1 &= \bar{d}_1 - \bar{d}_2 - L\bar{d}_3 - e\bar{d}_4 + \frac{qL^2}{2}\left[\frac{1}{GA_2} + \frac{e^2}{GJ} + \frac{L^2}{4EI_3}\right] \\ &\quad - \int_0^L k_3^0 (L - x_1) dx_1 \end{aligned} \qquad \text{(e)}$$

The value of Z_1 for no initial strain or support movement is

$$Z_1 = \frac{3}{8} qL \left[\frac{1 + \dfrac{4E}{G}\left(\dfrac{I_3}{A_2L^2} + \dfrac{e^2 I_3}{JL^2}\right)}{1 + \dfrac{3E}{G}\left(\dfrac{I_3}{A_2L^2} + \dfrac{e^2 I_3}{JL^2}\right)}\right] \qquad \text{(f)}$$

Final Forces

The total forces are obtained by superimposing the forces due to the prescribed external system and the redundants:

$$\begin{aligned} F_2 &= F_{2,0} + Z_1 F_{2,1} = -q(L - x_1) + Z_1 \\ M_T &= qe(L - x_1) - eZ_1 \\ M_3 &= -\frac{q}{2}(L - x_1)^2 + (L - x_1)Z_1 \\ R_1 &= Z_1 \\ R_2 &= qL - Z_1 \\ R_3 &= \frac{qL^2}{2} - LZ_1 \\ R_4 &= e(qL - Z_1) \end{aligned} \qquad \text{(g)}$$

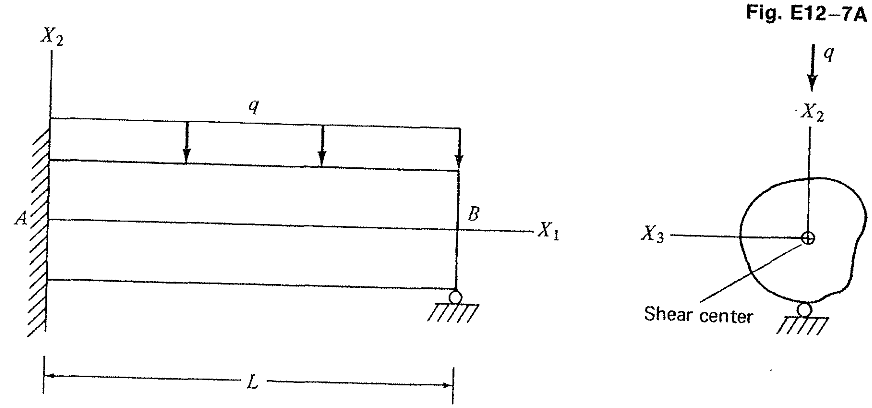
Example 12–7

This loading (Fig. E12–7A) will produce only flexure in the X_1-X_2 plane. We suppose the material is physically *nonlinear* and take the expression for k_3 as

$$k_3 = k_3^0 + a_1 M_3 + a_3 M_3^3 \qquad \text{(a)}$$

To simplify the analysis, we neglect transverse shear deformation.

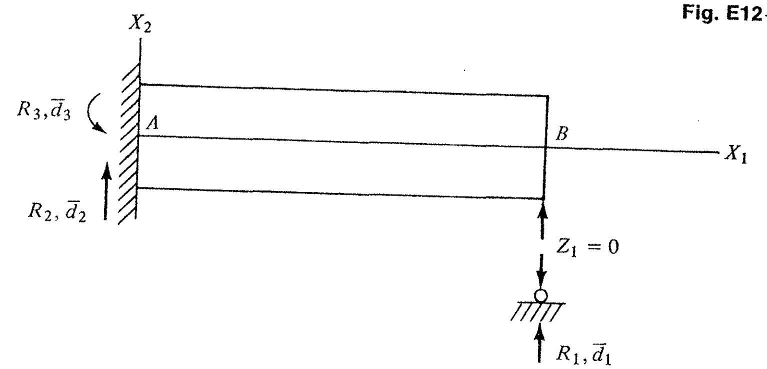
Fig. E12–7A



Primary Structure

$$R_1 = Z_1 \qquad R_2 = -(F_2)_{x_1=0} \qquad R_3 = -(M_3)_{x_1=0} \qquad \text{(b)}$$

Fig. E12–7B



Force System Due to Prescribed External Forces (see Example 12–6)

$$\begin{aligned} F_{2,0} &= -q(L - x_1) \\ M_{3,0} &= -\frac{q}{2}(L - x_1)^2 \end{aligned} \qquad \text{(c)}$$

$$R_{1,0} = 0 \qquad R_{2,0} = qL \qquad R_{3,0} = \frac{qL^2}{2}$$

Force System Due to $Z_1 = +1$ (see Example 12–6)

$$F_{2,1} = +1 \qquad M_{3,1} = L - x_1$$
$$R_{1,1} = +1 \qquad R_{2,1} = -1 \tag{d}$$
$$R_{3,1} = -L$$

Compatibility Equation

Since the material is nonlinear, we must use (12–33). Neglecting the transverse shear deformation term (e_2), the compatibility condition reduces to

$$\int_0^L k_3 M_{3,1}\, dx_1 = \sum \bar{d}_i R_{i,1} \tag{e}$$

We substitute for k_3 using (a):

$$\int_0^L (a_1 M_3 + a_3 M_3^3) M_{3,1}\, dx_1 = \sum \bar{d}_i R_{i,1} - \int_0^L k_3^0 M_{3,1}\, dx_1 \tag{f}$$

Now,

$$M_3 = M_{3,0} + Z_1 M_{3,1} \tag{g}$$
$$= -\frac{q}{2}(L - x_1)^2 + Z_1(L - x_1)$$

Introducing (g) in (f), we obtain the following cubic equation for Z_1:

$$Z_1^3\left(\frac{a_3 L^5}{5}\right) + Z_1^2\left(-\frac{a_3 q L^6}{4}\right) + Z_1\left(\frac{a_1 L^3}{3} + \frac{3a_3 q^2 L^7}{28}\right)$$
$$= \frac{qL^4}{8}\left(a_1 + \frac{a_3 q^2 L^4}{8}\right) + \bar{d}_1 - d_2 - L\bar{d}_3 - \int_0^L k_3^0 (L - x_1)\,dx_1 \tag{h}$$

For the physically linear case,

$$a_1 = \frac{1}{EI_3} \qquad a_3 = 0 \tag{i}$$

and (h) reduces to

$$Z_1 = \frac{3}{8}qL + \frac{3EI_3}{L^3}\left[\bar{d}_1 - \bar{d}_2 - L\bar{d}_3 - \int_0^L k_3^0 (L - x_1)\,dx_1\right] \tag{j}$$

Example 12–8

The member shown (Fig. E12–8A) is fixed at both ends. We consider the case where the material is linearly elastic, and there are no support movements or initial strains. We take the end actions at B referred to the *shear center* as the force redundants.

$$Z_1 = \bar{F}_{B2}$$
$$Z_2 = \bar{M}_{B3} \tag{a}$$
$$Z_3 = \bar{M}_{TB}$$

The forces acting on the primary structure are shown in Fig. E12–8B.

Initial Force System

$$F_{2,0} = P \qquad M_{3,0} = P(a - x_1) \tag{b}$$
$$M_{T,0} = P\bar{x}_3$$

Fig. E12–8A

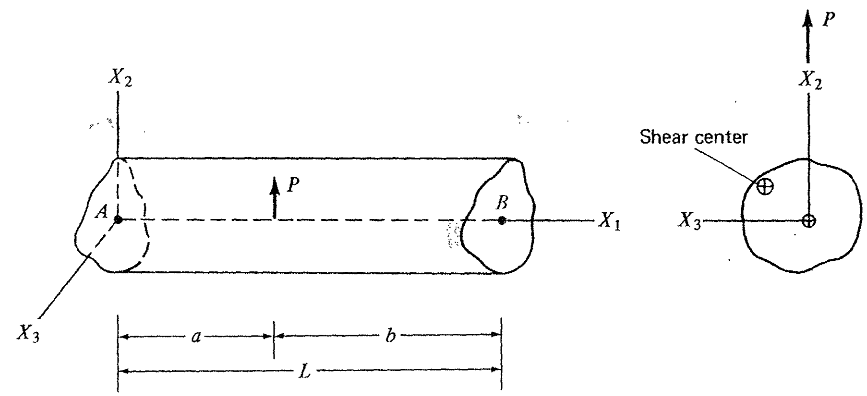


Fig. E12–8B

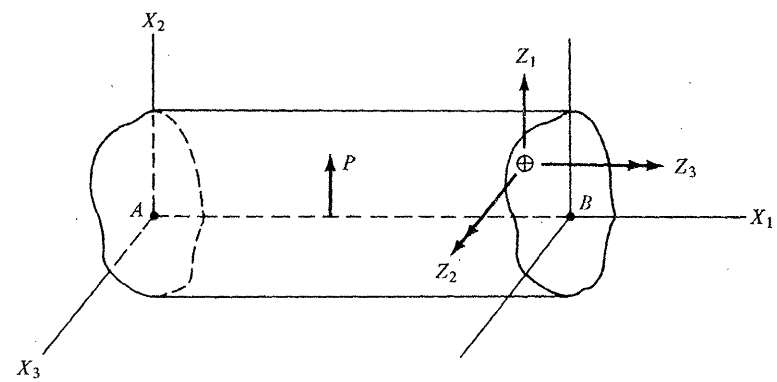
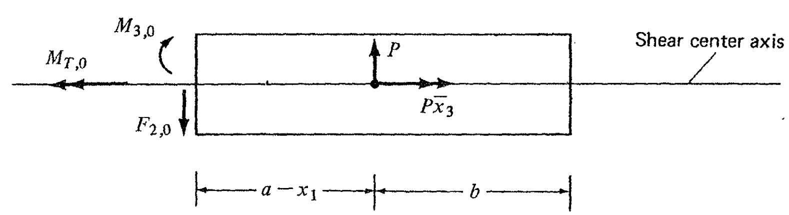
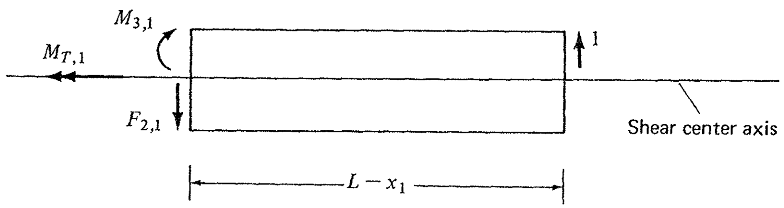


Fig. E12–8C



$Z = +1$

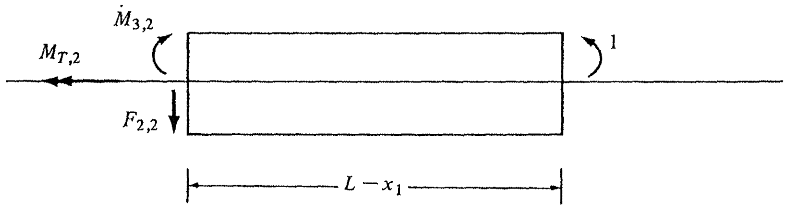
Fig. E12–8D



$$F_{2,1} = +1 \qquad M_{3,1} = L - x_1 \qquad \text{(c)}$$
$$M_{T,1} = 0$$

$Z_2 = +1$

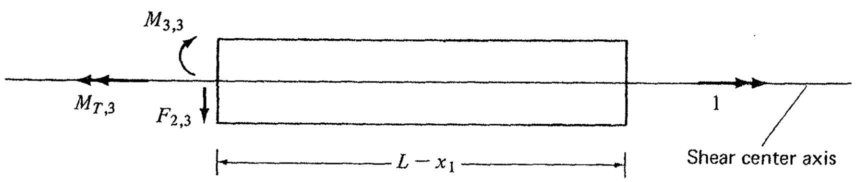
Fig. E12–8E



$$M_{3,2} = +1 \qquad F_{2,2} = M_{T,2} = 0 \qquad \text{(d)}$$

$Z_3 = +1$

Fig. E12–8F



$$M_{T,3} = +1 \qquad F_{2,3} = M_{3,3} = 0 \qquad \text{(e)}$$

Compatibility Equations

The compatibility equations for this problem have the form

$$\sum_{j=1}^{3} f_{kj} Z_j = \Delta_k \qquad (k = 1, 2, 3)$$

$$f_{kj} = \int_0^L \left[\frac{1}{GA_2} F_{2,j} F_{2,k} + \frac{1}{GJ} M_{T,j} M_{T,k} + \frac{1}{EI_3} M_{3,j} M_{3,k} \right] dx_1 \qquad \text{(f)}$$
$$\Delta_k = -\int_0^a \left[\left(\frac{F_{2,0}}{GA_2}\right) F_{2,k} + \left(\frac{M_{T,0}}{GJ}\right) M_{T,k} + \left(\frac{M_{3,0}}{EI_3}\right) M_{3,k} \right] dx_1$$

Substituting for the various forces and evaluating the resulting integrals lead to the following equations:

$$\left(\frac{L}{GA_2} + \frac{L^3}{3EI_3}\right) Z_1 + \left(\frac{L^2}{2EI_3}\right) Z_2 = -P\left[\frac{a}{GA_2} + \frac{1}{EI_3}\left(\frac{a^3}{3} + \frac{a^2 b}{2}\right)\right]$$
$$\left(\frac{L^2}{2EI_3}\right) Z_1 + \left(\frac{L}{EI_3}\right) Z_2 = -\frac{Pa^2}{2EI_3} \qquad \text{(g)}$$
$$\left(\frac{L}{GJ}\right) Z_3 = -\frac{Pa\bar{x}_3}{GJ}$$

Finally, solving (g), we obtain

$$Z_1 = -P\left(\frac{a}{L}\right)^2 \left[1 + \frac{2b}{L} \frac{1 + \dfrac{6EI_3}{aLGA_2}}{1 + \dfrac{12EI_3}{L^2 GA_2}}\right] \qquad \text{(h)}$$
$$Z_2 = P\frac{a^2 b}{L^2} \left[\frac{1 + \dfrac{6EI_3}{aLGA_2}}{1 + \dfrac{12EI_3}{L^2 GA_2}}\right] \qquad Z_3 = -\frac{Pa\bar{x}_3}{L}$$

Application

Suppose the member is subjected to the distributed loading shown in Fig. E12–8G. We can determine the force redundants by substituting for P, a, and b in (h),

$$P = q\,dx_1$$
$$a = x_1 \qquad \text{(i)}$$
$$b = L - x_1$$

and integrating the resulting expressions. The general solution is

$$Z_1 = \frac{-1}{L^2} \int_0^L \left\{ x_1^2 + \frac{2}{LC}\left[x_1^2(L - x_1) + \frac{6EI_3}{LGA_2} x_1(L - x_1)\right]\right\} q\,dx_1$$
$$Z_2 = \frac{1}{L^2 C} \int_0^L \left[x_1^2(L - x_1) + \frac{6EI_3}{LGA_2} x_1(L - x_1)\right] q\,dx_1 \qquad \text{(j)}$$
$$Z_3 = -\frac{\bar{x}_3}{L} \int_0^L x_1 q\,dx_1$$

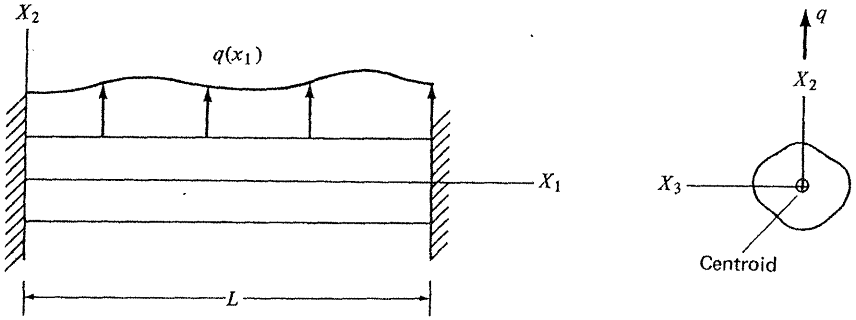
where

$$C = 1 + \frac{12EI_3}{L^2GA_2}$$

As an illustration, we consider the case where q is constant. Taking q = const in (j), we obtain

$$Z_1 = -\frac{qL}{2}$$
$$Z_2 = \frac{qL^2}{12} \qquad \text{(k)}$$
$$Z_3 = -\frac{\bar{x}_3 qL}{2}$$

Fig. E12–8G



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PROBLEMS

12–1. The accompanying sketch shows a sandwich beam consisting of a core and symmetrical face plates. The distribution of normal stress over the depth is determined by assuming a *linear* variation for the extensional strain:

$$\varepsilon_1 = -x_2 k_3$$
$$\sigma_{11} = E\varepsilon_1 \qquad \text{(a)}$$

We relate k_3 to M_3 by substituting for σ_{11} in the definition equation for M_3:

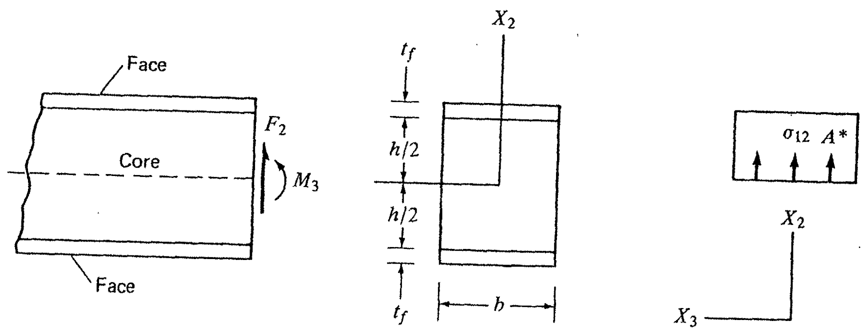
$$M_3 = -\iint_A x_2\sigma_{11}\, dA$$
$$\Downarrow \qquad \text{(b)}$$
$$M_3 = (E_c I_{3,c} + E_f I_{3,f})k_3$$

To simplify the notation, we drop the subscript and write (b) as

$$M = (EI)_{\text{equiv}} k_3 \qquad \text{(c)}$$

where $(EI)_{\text{equiv}}$ is the equivalent *homogeneous* flexural rigidity.

Prob. 12–1



The shearing stress distribution is determined by applying the engineering theory developed in Sec. 11–7. Integrating the axial force-equilibrium equation over the area A^* and assuming σ_{12} is constant over the width, we obtain

$$\iint_{A^*} (\sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3})dA = 0$$
$$\Downarrow \qquad \text{(d)}$$
$$b\sigma_{12} = \iint_{A^*} \sigma_{11,1}\, dA$$

Then, substituting for σ_{11},

$$\sigma_{11} = -(Ek_3)x_2 = \frac{M}{(EI)_{\text{equiv}}}(-Ex_2) \qquad \text{(e)}$$

and noting that $F_2 = -M_{3,1}$, (d) becomes

$$\sigma_{12} = \frac{F_2}{b(EI)_{\text{equiv}}} \iint_{A^*} x_2 E\, dA \qquad \text{(f)}$$

(a) Apply Equations (e) and (f) to the given section.
(b) The flange thickness is small with respect to the core depth for a typical beam. Also, the core material is relatively soft, i.e., E_c and G_c are small with respect to E_f. Specialize part a for $E_c = 0$ and $t_f/h \ll 1$. Also determine the equivalent shear rigidity $(GA_2)_{\text{equiv}}$, which is defined as

$$(\bar{V}^*)_{\sigma_{12}} = \iint_A \frac{\sigma_{12}^2}{2G}\, dA \equiv \frac{1}{2}\frac{F_2^2}{(GA_2)_{\text{equiv}}}$$

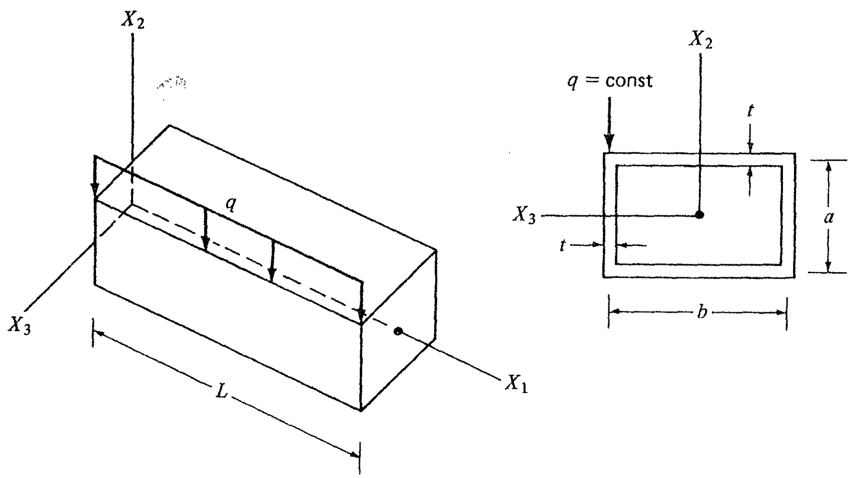
(c) The member force-deformation relations are

$$\gamma_2 = \frac{F_2}{(GA_2)_{\text{equiv}}}$$
$$k_3 = \frac{M_3}{(EI)_{\text{equiv}}}$$

Refer to Example 12–1. Specialize Equation (q) for this section and discuss when transverse shear deformation has to be considered.

12–2. Using the displacement method, determine the complete solution for the problem presented in the accompanying sketch. Comment on the influence of transverse shear deformation.

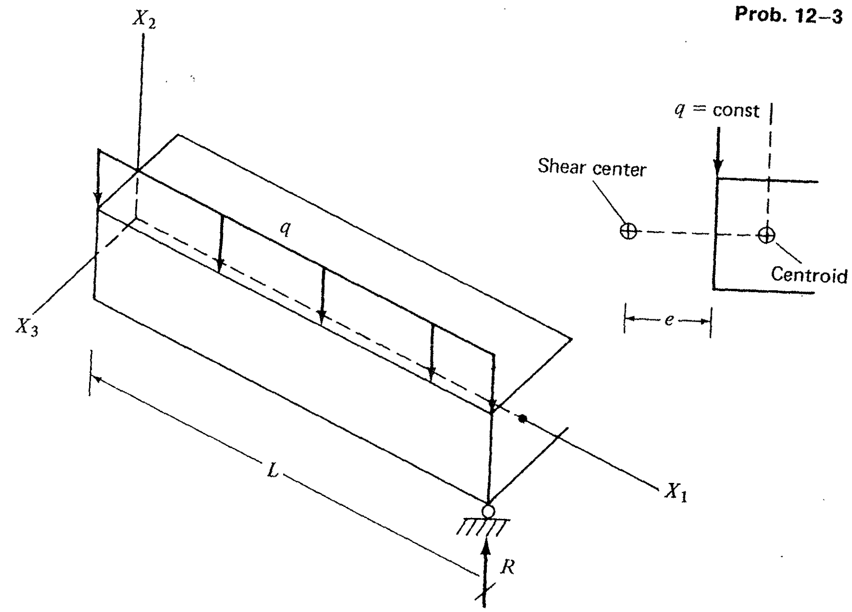
Prob. 12–2



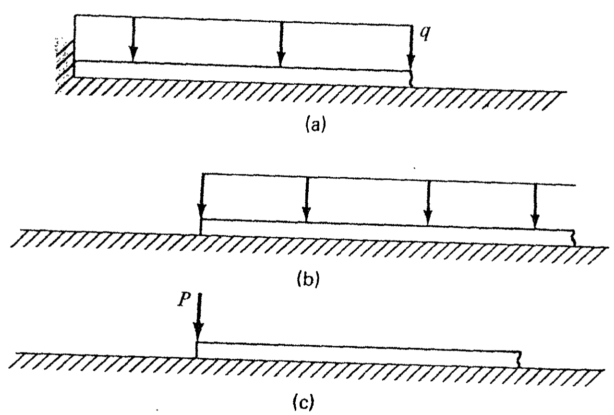
12–3. For the problem sketched, determine the complete solution by the displacement method.

12–4. Determine the solution for the cases sketched. Express the solution in terms of the ψ functions defined by (12–26).

Prob. 12–3



Prob. 12–4



12–5. The formulation for the beam on an elastic foundation is based on a continuous distribution of stiffness; i.e., we wrote

$$b_2 = -ku_2 \tag{a}$$

Note that k has units of force/(length)^2.

We can apply it to the system of discrete restraints diagrammed in part a of the accompanying sketch, provided that restraint spacing c is small in comparison to characteristic length (boundary layer) L_b, which we have taken as

$$L_b \approx \frac{3}{\lambda} = \frac{3}{(k/4EI)^{1/4}} \tag{b}$$

A reasonable upper limit on c is

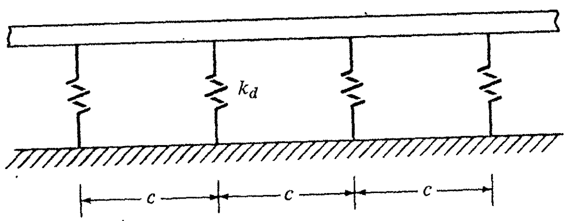
$$c < \approx \frac{L_b}{15} \tag{c}$$

Letting k_d denote the discrete stiffness, we determine the equivalent distributed stiffness k from

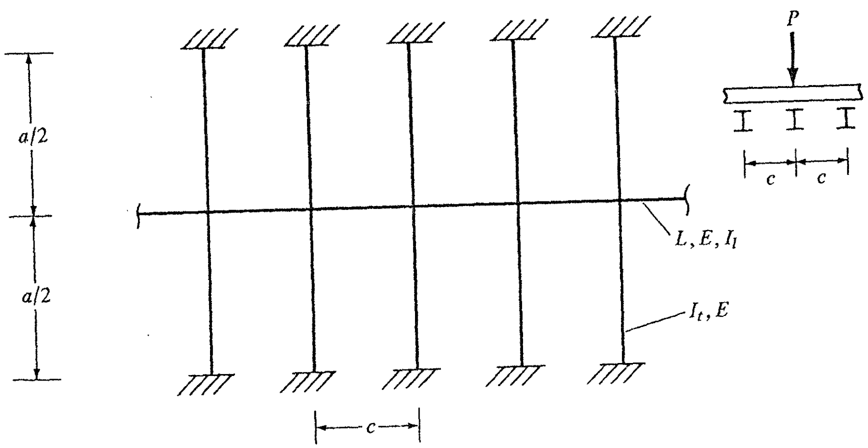
$$k = k_d/c \tag{d}$$

Evaluate L_b with (b), and then check c with (c).

Prob. 12–5



(a)



(b)

Consider the beam of part b, supported by cross members which are fixed at their ends. Following the approach outlined above, determine the distribution of force applied to the cross members due to the concentrated load, P.

Evaluate this distribution for

$$a = 24 \text{ ft} \qquad L = 64 \text{ ft} \qquad c = 1 \text{ ft} \qquad I_t = I_\ell$$

12–6. Refer to Example 12–3. The governing equation for a prismatic beam on a linearly elastic foundation with transverse shear deformation included is obtained by setting $b_2 = q - ku_2$ in (i). For convenience, we drop the subscripts:

$$\frac{d^4u}{dx^4} - \frac{k}{GA}\frac{d^2u}{dx^2} + \frac{k}{EI}u = \frac{1}{EI}\left(q - \frac{dm}{dx}\right) + \frac{d^2}{dx^2}\left(k^0 - \frac{q}{GA}\right) \tag{a}$$

We let

$$\frac{k}{EI} = 4\lambda^4 \qquad \frac{k}{GA_2} = 4\xi\lambda^2 \tag{b}$$

and (a) takes the form

$$\frac{d^4u}{dx^4} - 4\xi\lambda^2\frac{d^2u}{dx^2} + 4\lambda^4 u = \bar{q} \tag{c}$$

Note that ξ is dimensionless and λ has units of 1/length. The homogeneous solution is

$$u = e^{-ax}(C_1 \cos bx + C_2 \sin bx) + e^{+ax}(C_3 \cos bx + C_4 \sin bx)$$

where (d)

$$a = \lambda(1 + \xi)^{1/2}$$
$$b = \lambda(1 - \xi)^{1/2}$$

To specialize (d) for negligible transverse shear deformation, we set $\xi = 0$.

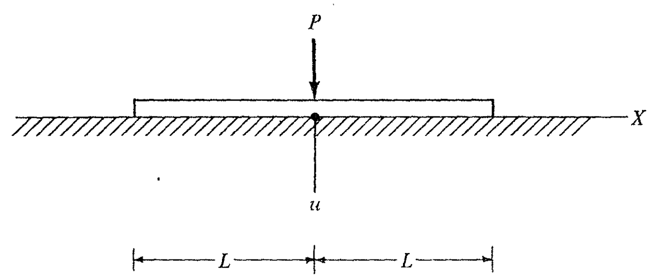
(a) Determine the expression for the boundary layer length ($e^{-3} \approx 0$).

(b) Determine the solution for the loading shown. Assume L large with respect to L_b. The boundary conditions at $x = 0$ are

$$\omega = 0$$
$$F_2 = -\frac{P}{2}$$

Investigate the variation of M_{max} and u_{max} with ξ. Consider ξ to vary from 0 to 1.

Prob. 12–6



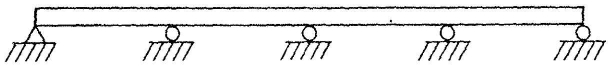
12–7. Refer to the sketch for Prob. 12–3. Determine the reaction R and centroidal displacements at $x_1 = L/2$ due to a concentrated force $P\bar{i}_2$ applied to the web at $x_1 = L/2$. Employ the force method.

12–8. Refer to Example 12–7. Assuming Equation (h) is solved for Z_1, discuss how you would determine the translation u_2 at $x_1 = L/2$.

12–9. Consider the four-span beam shown. Assume linearly elastic behavior, the shear center coincides with the centroid, and planar loading.

(a) Compare the following choices for the force redundants with respect to computational effort:
   1. reactions at the interior supports
   2. bending moments at the interior supports

(b) Discuss how you would employ Maxwell's law of reciprocal deflections to generate influence lines for the redundants due to a concentrated force moving from left to right.



Prob. 12–9

12–10. Consider a linearly elastic member fixed at both ends and subjected to a temperature increase

$$T = a_1 + a_2x_2 + a_3x_3$$

Determine the end actions and displacements (translations and rotations) at mid-span.

12–11. Consider a linearly elastic member fixed at the left end (A) and subjected to forces acting at the right end (B) and support movement at A. Determine the expressions for the displacements at B in terms of the support movement at A and end forces at B with the force method. Compare this approach with that followed in Example 12–2.

13

Restrained Torsion-Flexure of a Prismatic Member

13–1. INTRODUCTION

The engineering theory of prismatic members developed in Chapter 12 is based on the assumption that the effect of variable warping of the cross section on the normal and shearing stresses is negligible, i.e., the stress distributions predicted by the St. Venant theory, which is valid only for constant warping and no warping restraint at the ends, are used. We also assume the cross section is rigid with respect to in-plane deformation. This leads to the result that the cross section *twists* about the *shear center*, a fixed point in the cross section. Torsion and flexure are *uncoupled* when one works with the torsional moment about the shear center rather than the centroid. The complete set of governing equations for the engineering theory are summarized in Sec. 12–4.

Variable warping or warping restraint at the ends of the member leads to additional normal and shearing stresses. Since the St. Venant normal stress distribution satisfies the definition equations for F_1, M_2, M_3 identically, the additional normal stress, σ^r, must be statically equivalent to zero, i.e., it must satisfy

$$\iint \sigma^r_{11}\, dA = \iint x_2\sigma^r_{11}\, dA = \iint x_3\sigma^r_{11}\, dA = 0 \qquad (13\text{–}1)$$

The St. Venant flexural shear flow distribution is obtained by applying the engineering theory developed in Sec. 11–7. This distribution is statically equivalent to F_2, F_3 acting at the shear center. It follows that the additional shear stresses, σ^r_{12} and σ^r_{13}, due to warping restraint must be statically equivalent to only a torsional moment:

$$\begin{aligned} \iint \sigma^r_{12}\, dA &= 0 \\ \iint \sigma^r_{13}\, dA &= 0 \end{aligned} \qquad (13\text{–}2)$$

To account for warping restraint, one must modify the torsion relations. We will still assume the cross section is rigid with respect to in-plane deformation.