1.571 Structural Analysis and Control
Prof. Connor
Section 3: Analysis of Cable Supported Structures

Many high performance structures include cables and cable systems. Analysis and design of cable systems is a complex topic. Individual cables have a non-linear equivalent stiffness. In addition, the interaction between multiple cables must be considered. In order to be able to analyze and design cable supported structures, a number of different topics must be covered.

The topics to be covered are:

3.1 - Cable equations

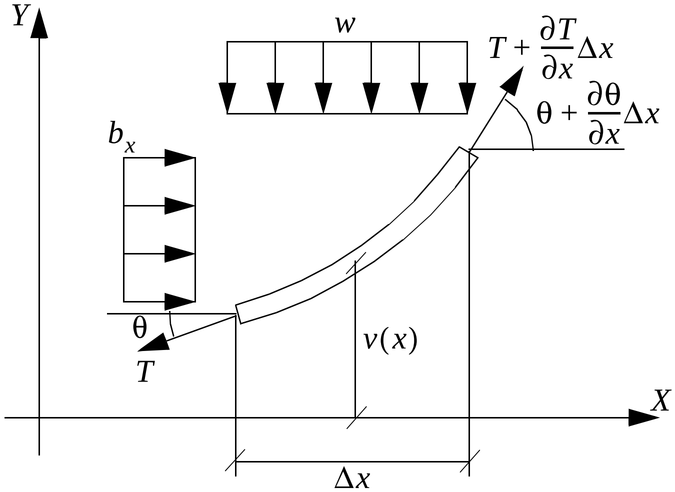
3.2 - Modeling of beam with single cable using an equivalent spring

3.3 - Modeling of beam with multiple cable using a beam on equivalent elastic foundation

3.4 - Design procedures for cable/beam system

3.1 Equations of a single cable

3.1.1 Equilibrium Equations



$$\sum F_x = -T\cos\theta + T\cos\theta + \frac{\partial}{\partial x}(T\cos\theta)\Delta x + b_x\Delta x = 0$$

$$\frac{\partial}{\partial x}(T\cos\theta) + b_x = 0$$

$$\sum F_y = -T\sin\theta + T\sin\theta + \frac{\partial}{\partial x}(T\sin\theta)\Delta x - w\Delta x = 0$$

$$\frac{\partial}{\partial x}(T\sin\theta) - w = 0$$

When $b = 0$;

$$\frac{\partial}{\partial x}(T\cos\theta) = 0 \rightarrow T\cos\theta = H = \text{constant}$$

For self-weight

γ = weight of cable / unit length

$$w\Delta x = \gamma\Delta s$$

$$wdx = \gamma ds$$

$$ds = \frac{dx}{\cos\theta}$$

$$w = \gamma\frac{dx}{dx\cos\theta} = \gamma\frac{1}{\cos\theta}$$

$$\frac{\partial}{\partial x}(T\sin\theta) - \frac{\gamma}{\cos\theta} = 0$$

$$T\sin\theta = \frac{H}{\cos\theta}\sin\theta = h\tan\theta$$

$$\tan\theta = \frac{dv}{dx}$$

So

$$\frac{\partial}{\partial x}\left(H\frac{dv}{dx}\right) - \frac{\gamma}{\cos\theta} = 0$$

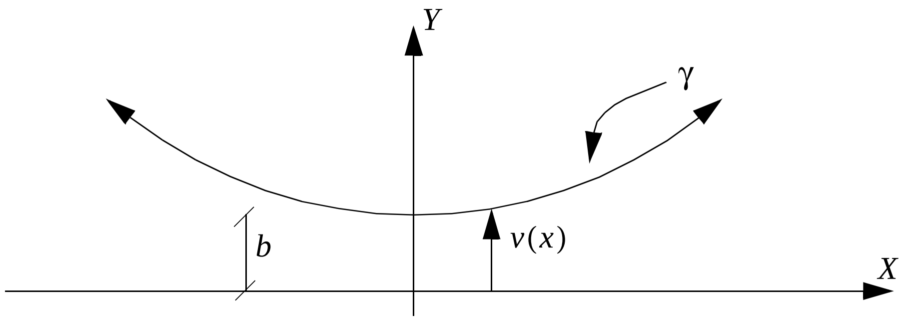
Knowing

$$\cos\theta = \frac{1}{\alpha} = \frac{1}{\sqrt{1+\left(\frac{dv}{dx}\right)^2}}$$

we get

$$\frac{d^2v}{dx^2} = \frac{\gamma}{H}\sqrt{1+\left(\frac{dv}{dx}\right)^2}$$

3.1.2 Example



Boundary Conditions

$$\left.\frac{dv}{dx}\right|_0 = 0 \qquad\qquad v(0) = b$$

Solution

Let $\frac{dv}{dx} = p$, $\frac{d^2v}{dx^2} = \frac{dp}{dx}$, and $\phi = \frac{\gamma}{H}$

so

$$\frac{dp}{dx} = \phi\sqrt{1+p^2}$$

$$\int\frac{dp}{\sqrt{1+p^2}} = \int\phi dx$$

Integration by parts leads to

$$\ln(p+\sqrt{1+p^2}) = \phi x + C_1$$

$$p + \sqrt{1 + p^2} = e^{\phi x + C_1} = e^{C_1} e^{\phi x} = C_2 e^{\phi x}$$

for

$$\frac{dv}{dx} = p = 0 \text{ at } x = 0$$

$$C_2 = 1$$

Then

$$p + \sqrt{p^2 + 1} = e^{\phi x}$$

$$\sqrt{p^2 + 1} = e^{\phi x} - p$$

$$p^2 + 1 = e^{2\phi x} - 2pe^{\phi x} + p^2$$

$$2p = \left(\frac{e^{2\phi x} - 1}{e^{\phi x}}\right) = e^{\phi x} - e^{-\phi x}$$

$$p = \frac{dv}{dx} = \frac{1}{2}(e^{\phi x} - e^{-\phi x})$$

Integrating

$$v = \frac{1}{2\phi}(e^{\phi x} + e^{-\phi x}) + C_3$$

for $y = b$ at $x = 0$

$$C_3 = b - \frac{1}{\phi} = b - \frac{H}{\gamma}$$

Setting $b = \frac{H}{\gamma}$ (for convenience)

$$C_3 = 0$$

Finally

$$v = \frac{H}{2\gamma}\left(e^{\frac{\gamma}{H}x} + e^{-\frac{\gamma}{H}x}\right)$$

$$v = \frac{H}{\gamma}\cosh\frac{\gamma}{H}x = \text{caternary (chain)}$$

at any point along the cable

$$T = \frac{H}{\cos\theta}$$

Then

$$T = \sqrt{1 + \left(\frac{dv}{dx}\right)^2} H$$

For w = constant and $b = 0$

$$T\cos\theta = H \rightarrow T = \frac{H}{\cos\theta}$$

$$\frac{\partial}{\partial x}(T\sin\theta) - w = 0$$

$$\frac{\partial}{\partial x}(H\tan\theta) - w = 0$$

$$\frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right) - w = 0$$

$$H\frac{\delta^2 v}{\delta x^2} = w$$

then $$Hv = C_1 + C_2 x + \frac{1}{2}wx^2$$

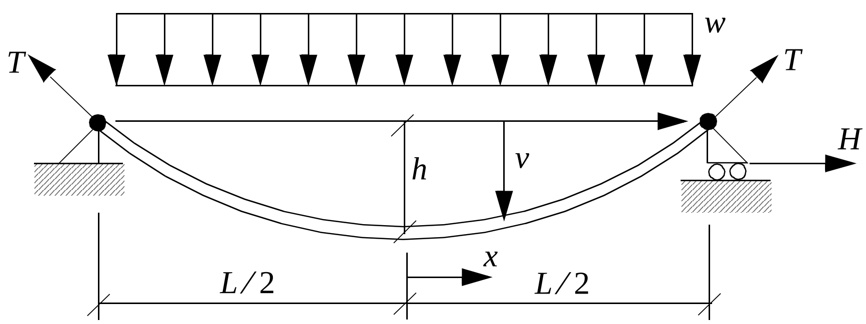
Boundary Conditions

$$x = x_1 \qquad v = v_1$$

$$x = x_2 \qquad v = v_2$$

3.1.3 More Examples

Example #1



$v = 0$ at $x = L/2$

$v,_x = 0$ at $x = 0$

$$Hv = C_1 + C_2 x + \frac{1}{2}wx^2$$

$$Hv,_x = C_2 + wx$$

then

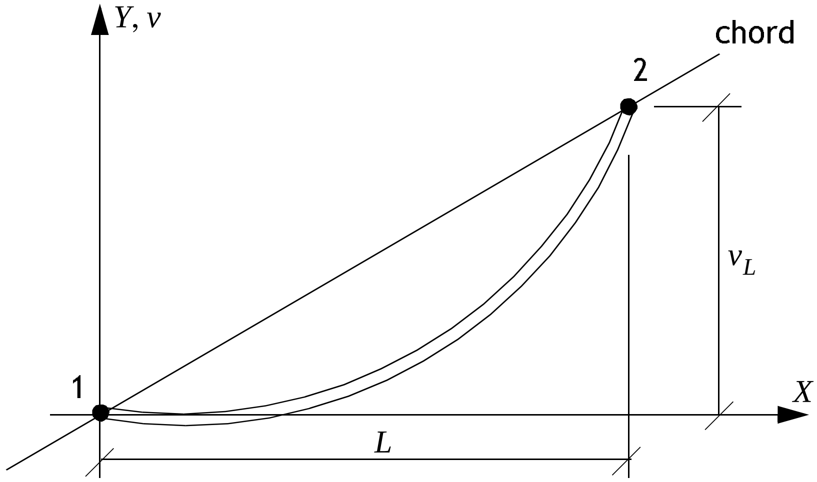
$$C_2 = 0$$

$$C_1 = -\frac{1}{2}\left(\frac{L}{2}\right)^2 w$$

$$v(x) = \frac{1}{2H}w\left(x^2 - \left(\frac{L}{2}\right)^2\right)$$

$$v(0) = v_{max} = \frac{wL^2}{8H} = h$$

Example #2



Point 1 at $x = 0$ $v_1 = 0$

Point 2 at $x = L$ at $v_2 = v_L$

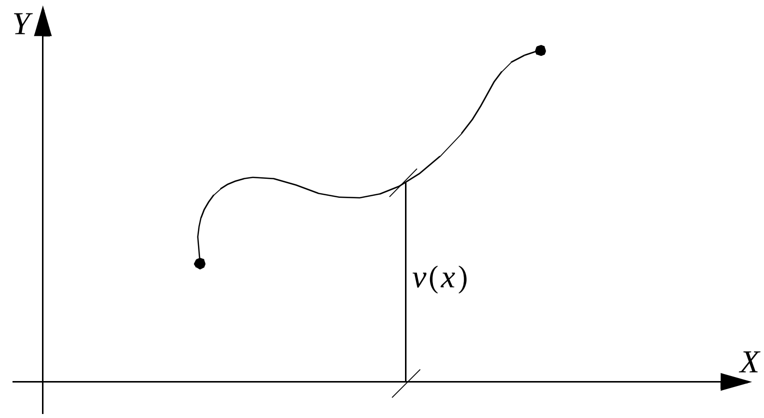
$$C_1 = 0$$

$$C_2 = \frac{1}{L}\left(Hv_L - \frac{1}{2}wL^2\right)$$

and

$$v(x) = \frac{x}{L}v_L + \underbrace{\frac{wL^2}{2H}\left(\left(\frac{x}{L}\right)^2 - \frac{x}{L}\right)}_{\text{deflection from chord}}$$

3.1.4 Geometric Relations - Arc Length



$$ds^2 = dx^2 + (v_{,x})^2 dx^2$$

$$ds = dx(1 + (v_{,x})^2)^{1/2}$$

$$s_{12} = \int_{s_1}^{s_2} ds = \int_{x_1}^{x_2} (1 + (v_{,x})^2)^{1/2} dx$$

Simplification for shallow curve

$(v_{,x})^2$ small wrt 1

So

$$\sin\theta \approx \tan\theta = v_{,x}$$

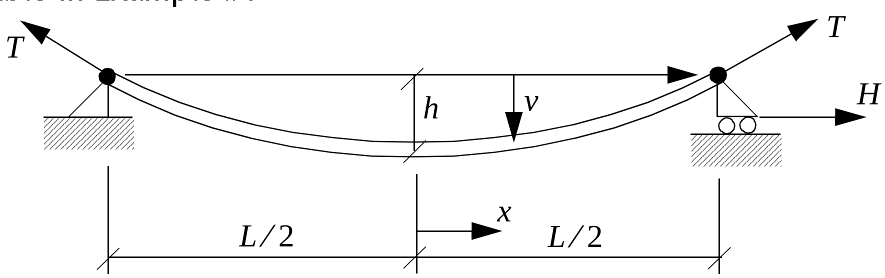
$$\cos\theta \approx 1$$

$$T\cos\theta \approx T$$

Then

$T \approx H$ = constant if there is only loading in the y-direction

Consider cable in Example #1



$$v(x) = -\frac{wL^2}{2H}\left(1 - \left(\frac{x}{L/2}\right)^2\right)$$

$$Hh = \frac{wL^2}{8}$$

Differentiate

$$v,_x = -\frac{wL^2}{8H}\left(-2\left(\frac{x}{L/2}\right)\frac{1}{L/2}\right)$$

$$v,_x = \frac{wx}{H} = \frac{w}{T}x$$

Approximate s (for shallow curve)

$$s_{12} \approx \int_{x_1}^{x_2} (1 + (v,_x)^2)^{1/2} dx \approx \int_{x_1}^{x_2} \left(1 + \frac{1}{2}(v,_x^2)\right) dx$$

$$\frac{s}{2} \approx \int_0^{L/2} \left(1 + \frac{1}{2}\left(\frac{wx}{H}\right)^2\right) dx$$

$$\frac{s}{2} \approx \left(x + \frac{1}{6}\left(\frac{w}{H}\right)^2 x^3\right)\Bigg|_0^{L/2}$$

$$\frac{s}{2} \approx \frac{L}{2} + \frac{1}{6}\left(\frac{w}{H}\right)^2\left(\frac{L}{2}\right)^3 = \frac{L}{2} + \frac{1}{6}\left(\frac{w}{H}\right)^2 \frac{L^3}{8}$$

$$s \approx L + \frac{1}{24}\left(\frac{w}{H}\right)^2 L^3$$

$$H = \frac{wL^2}{8h}$$

$$\boxed{s \approx L + \frac{1}{24}\left(\frac{8hw}{wL^2}\right)^2 L^3 \approx L + \frac{8}{3}\frac{h^2}{L} \approx L\left\{1 + \frac{8}{3}\left(\frac{h}{L}\right)^2\right\}}$$

Note:

s = deformed length

s_o = initial length

The "shallow" assumption implies $T \approx H$ = constant

$$\varepsilon = \frac{T}{AE}$$

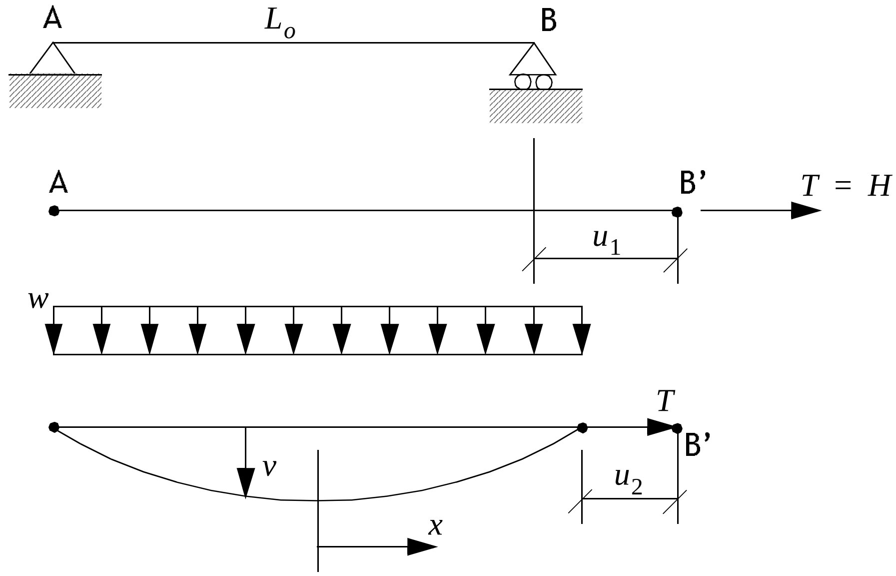
$$\Delta s = \varepsilon s_o = \frac{T}{AE}s_o$$

$$s \approx s_o + \Delta s = s_o\left(1 + \frac{T}{AE}\right) \approx s_o\left(1 + \frac{H}{AE}\right)$$

$$s_o\left(1 + \frac{H}{AE}\right) = L\left(1 + \frac{1}{24}\left(\frac{w}{H}\right)^2 L^2\right)$$

$$\boxed{\frac{s_o}{L} = \frac{\left(1 + \frac{1}{24}\left(\frac{w}{H}\right)^2 L^2\right)}{\left(1 + \frac{H}{AE}\right)}}$$

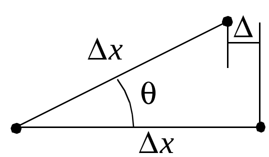
3.1.5 Equivalent Tangent Stiffness



$$u_1 = \frac{T}{AE}L_o \qquad u_2 = \frac{1}{2}\int_0^{L_o} v_{,x} dx$$

- Consider Cable AB with initial length L_o
- Apply tension T, which results in u_1
- Apply transverse loading, w, which results in a "negative" end movement u_2. u_2 is a function of T.
- The net movement is $u_1 - u_2 = u_B$.

Chord Shortening



$$\Delta = \Delta x - \Delta x \cos\theta = \Delta x(1 - \cos\theta)$$

for small θ $\qquad \cos\theta \approx 1 - \frac{\theta^2}{2}$

and $\qquad \theta \approx v_{,x}$

Then

$$\Delta \approx \Delta x\left(\frac{\theta^2}{2}\right) \approx \frac{\Delta x}{2}(v_{,x}^2)$$

$$u_2 = \int\Delta \approx \frac{1}{2}\int v_{,x}^2 dx$$

$$v_{,x} = \frac{w}{T}x$$

$$u_2 = 2\left(\frac{1}{2}\int_0^{L/2}\left(\frac{w}{T}x\right)^2 dx\right) = \left(\frac{w}{T}\right)^2 \frac{L_o^3}{24}$$

$$u_B = u_1 - u_2 = L_o\left\{\frac{T}{AE} - \frac{1}{24}\left(\frac{wL_o}{T}\right)^2\right\}$$

Perturbation

- Increment T by an amount ΔT
- Get corresponding change in u_B

$$u_B + \Delta u_B = L_o\left\{\frac{T + \Delta T}{AE} - \frac{1}{24}\left(\frac{wL_o}{T + \Delta T}\right)^2\right\}$$

For small ΔT wrt T

$$\Delta u_B \approx \frac{\partial u_B}{\partial T}\Delta T$$

$$du_B \approx \frac{\partial u_B}{\partial T}dT$$

$$\frac{\partial u_B}{\partial T} = L_o\left\{\frac{1}{AE} + \frac{1}{24}\frac{(wL_o)^2}{T^3}\right\}$$

$$du_B = L_o\left\{\frac{1}{AE} + \frac{1}{24}\frac{(wL_o)^2}{T^3}\right\}dT$$

$$f_B = \text{tangent flexibility} = \frac{du_B}{dT}$$

$$dT = \frac{1}{f_B}du_B = k_B du_B$$

$$k_B = \text{tangent stiffness}$$

$$k_B = \frac{AE/L_o}{1 + \frac{1}{12}\left(\frac{AE}{T}\right)\left(\frac{wL_o}{T}\right)^2}$$

Note: k_B approaches $\frac{AE}{L_o}$ as T increases

Write

$$k_B = \frac{A}{L_o}E_{eff}$$

$$E_{eff} = \text{effective modulus} = \frac{E}{1 + \frac{1}{12}\left(\frac{AE}{T}\right)\left(\frac{wL_o}{T}\right)^2}$$

Alternate forms (shallow cable)

$$Hh \approx Th = \frac{wL^2}{8} \rightarrow \frac{wL_o}{T} = 8\frac{h}{L_o}$$

$$T = A\sigma^*$$

where σ^* = intial cable stress

So

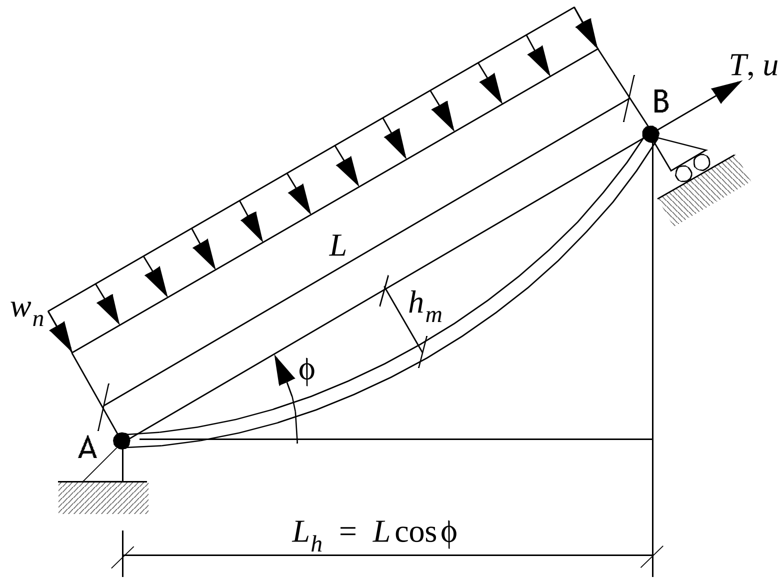
$$E_{eff} = \frac{E}{1 + \frac{1}{12}\frac{E}{\sigma^*}\left(\frac{8h}{L_o}\right)^2} = \frac{E}{1 + \frac{16}{3}\frac{E}{\sigma^*}\left(\frac{h}{L_o}\right)^2}$$

and

$$T = \frac{wL_o}{8(h/L_o)} = A\sigma^*$$

$$A_{cable} = \frac{wL_o}{8\sigma^*(h/L_o)}$$

3.1.6 Inclined Shallow Cable

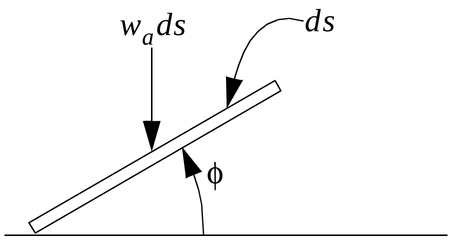


$$k = \frac{A}{L}E_{eff}$$

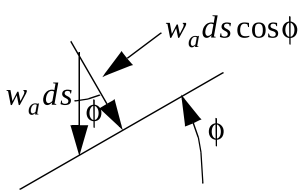
$$E_{eff} = \frac{E}{1 + \frac{1}{12}\left(\frac{AE}{T}\right)\left(\frac{w_n L}{T}\right)^2}$$

$$\frac{w_n L}{T} = 8\frac{h_m}{L}$$

Evaluation of w_n for self-weight



w_a = unit weight per unit length of center line



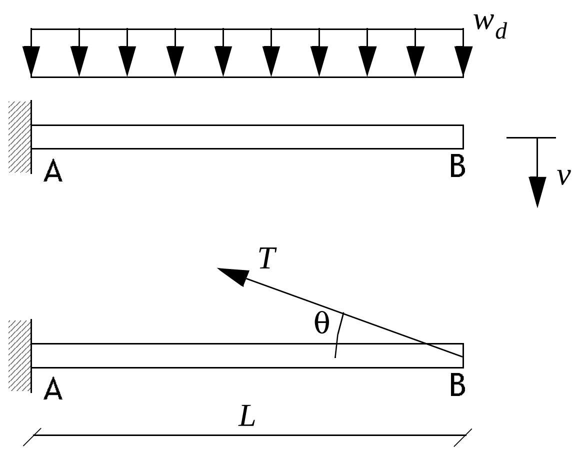
$$w_n = w_a\cos\phi$$

Then $$w_n L = (w_a\cos\phi)L = w_a(\cos\phi L) = w_a L_h$$

$$E_{eff} = \frac{E}{1 + \frac{1}{12}\left(\frac{AE}{T}\right)\left(\frac{w_a L_h}{T}\right)^2}$$

3.2 Modeling of beam with single cable using an equivalent spring

Example #1: Cantilever beam with single cable



Reaction at A due to w_d

$$M_A = -\frac{wdL^2}{2}$$

Deflection at B due to w_d

$$v_B = \frac{wdL^4}{8EI}$$

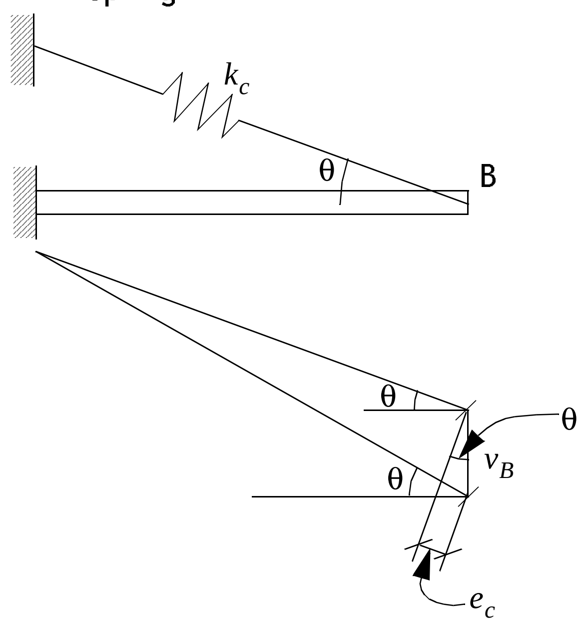
Reaction at A due to T

$$M_A = T\sin\theta L$$

Deflection at B due to T

$$v_B = \frac{T\sin\theta L^3}{3EI}$$

Beam and Spring Model



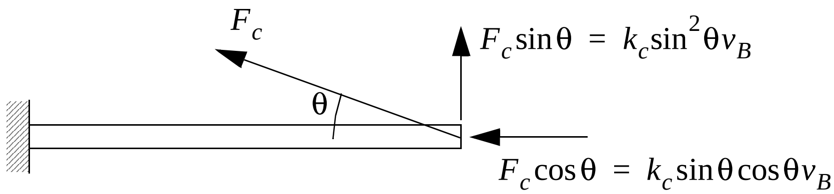
For small v_B

e_c = spring extension

$$e_c = v_B\sin\theta$$

F_c = incremental force in cable

$$F_c = k_c e_c = k_c \sin\theta v_B$$

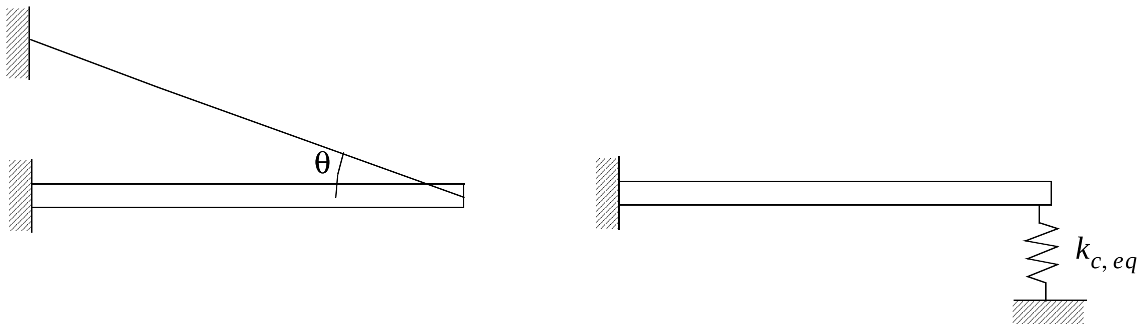


for pretensioned cable

$$k_c = \frac{A_c}{L_c} E_{eff}$$

$$E_{eff} = \frac{E}{1 + \frac{1}{12}\frac{AE}{T}\left(\frac{w_n L}{T}\right)^2}$$

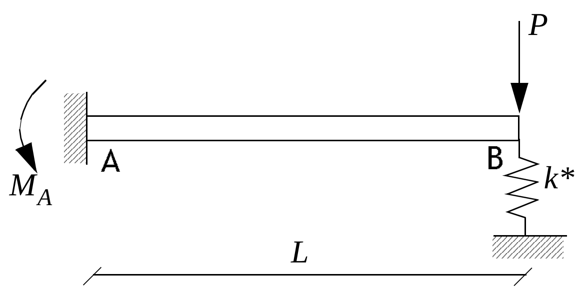
Non-Linear Spring Model



$$k_{c,\,eq} = k_c \sin^2\theta = k^*$$

This model is used to determine the “incremental” forces due to line loading.

Illustration



Deflection at B

$$v_B = v_{Bp} + v_{Bs} = \frac{PL^3}{3EI} - \frac{F_s L^3}{3EI}$$

$$\frac{(P-F)L^3}{3EI} = v_B = \frac{(P - k^* v_B)L^3}{3EI}$$

$$v_B\left(\frac{3EI}{L^3} + k^*\right) = P$$

Moment Reaction at A

$$M_A = (PL - k^*v_BL) = PL\left(1 - \frac{k^*v_B}{P}\right)$$

$$M_A = PL\left(1 - \frac{k^*}{\frac{3EI}{L^3} + k^*}\right)$$

$$M_A = PL\left(1 - \frac{1}{1 + \frac{3EI}{L^3}\frac{1}{k^*}}\right)$$

Note: Inclusion of cable reduces the negative moment at the support and also the deflection at the end point. This effect depends on the relative stiffness of the beam vs the cable.

$$M_A = PL(1 - \alpha)$$

$$\alpha = \frac{1}{1 + \frac{3EI}{L^3}\frac{1}{k^*}}$$

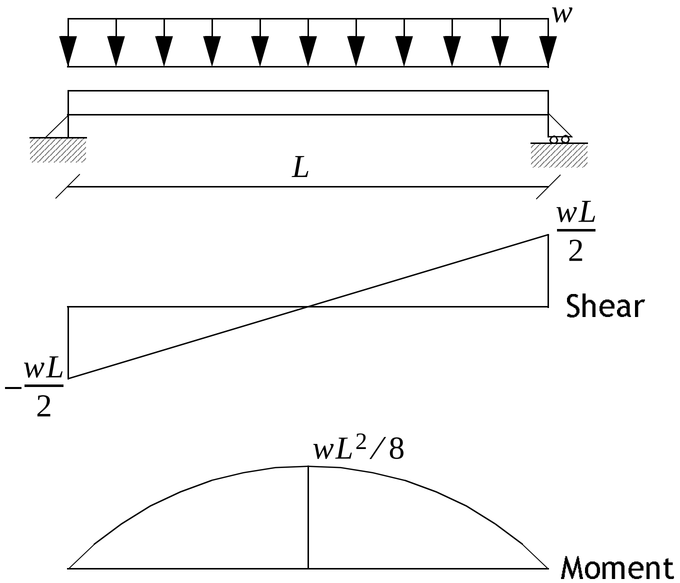
So, for

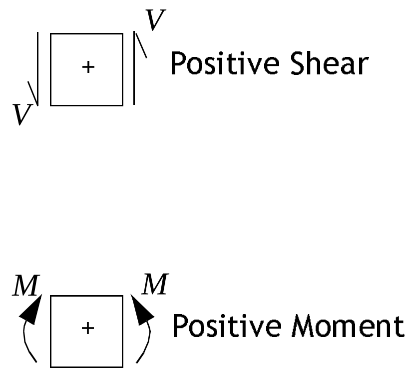
$k^* \approx 0$ $\quad \alpha = 0$ and $M_A = PL$

$k^* \approx \infty$ $\quad \alpha = 1$ and $M_A = 0$

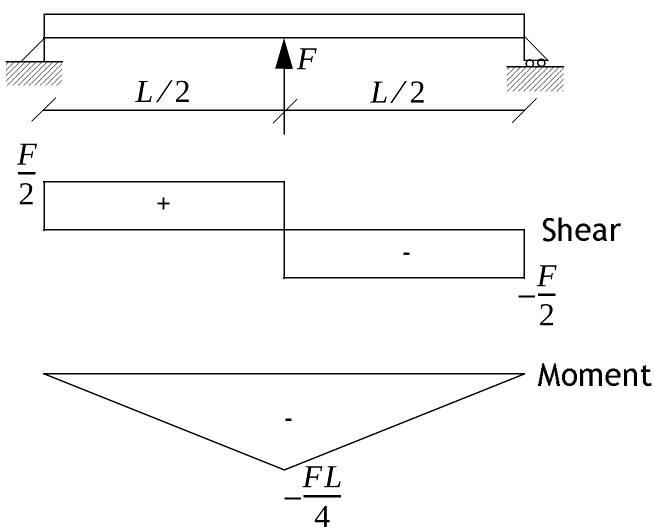
Example #2

Case a

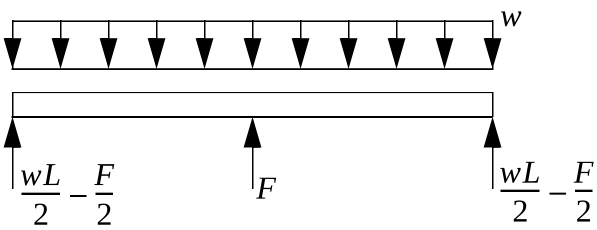




Case b



Case c



$$M = \left(\frac{wL}{2} - \frac{F}{2}\right)x - \frac{wx^2}{2}$$

$$V = \frac{F}{2} - \frac{wL}{2} + wx$$

Let

$$F = \alpha(wL)$$

Then

$$M = \frac{wL}{2}(1-\alpha)x - \frac{wx^2}{2}$$

$$V = \frac{wL}{2}(1-\alpha) + wx$$

M_{max} occurs at x such that $V(x^*) = 0$

$$x^* = \frac{L}{2}(1-\alpha)$$

$$M_{max} = M^* = \frac{wL^2}{8}(1-\alpha)^2$$

M_{min} occurs at $x = \frac{L}{2}$

$$M_{min} = \frac{FL}{4} - \frac{wL^2}{8} = \frac{wL^2}{8}(2\alpha - 1)$$

Optimum Case

$$|M^*| = |M(L/2)|$$

$$(1-\alpha)^2 = 2\alpha - 1$$

$$\alpha = 2 \pm \sqrt{2}$$

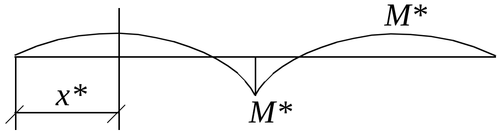
For $\alpha < 1$ (x^* must be positive)

$$\alpha = 2 - \sqrt{2} = 0.586$$

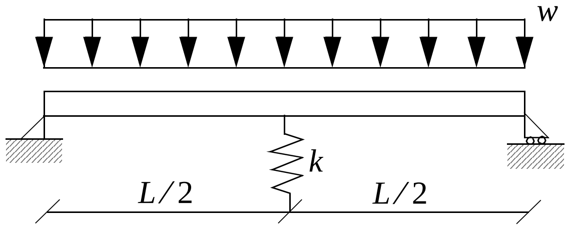
$$M|_{peak} = \frac{wL^2}{8}(0.172)$$

$$x^* = 0.414\frac{L}{2}$$

Case c (moment balanced)



Suppose F is a spring resultant



δ = deflection at $x = L/2$ (↓ +ve)

$$\delta = \delta_w + \delta_s$$

$$\delta_w = \frac{5wL^4}{384EI}$$

$$\delta_s = -\frac{F_s L^3}{48EI} = -\frac{k\delta L^3}{48EI}$$

$$\delta = \frac{5wL^4}{384EI} - \frac{k\delta L^3}{48EI}$$

$$\delta = \frac{5wL}{8}\left(\frac{1}{\frac{48EI}{L^3} + k}\right)$$

Then

$$F = \delta k = wL\frac{5}{8}\left(\frac{1}{\frac{48EI}{L^3} + k}\right) = \alpha wL$$

Express k in terms of α

$$k = \frac{48EI}{L^3}\left(\frac{1}{\frac{5}{8\alpha} - 1}\right)$$

$\alpha = \frac{5}{8} \rightarrow k = \infty$ Rigid Support

$\alpha = 0 \rightarrow k = 0$ No Support

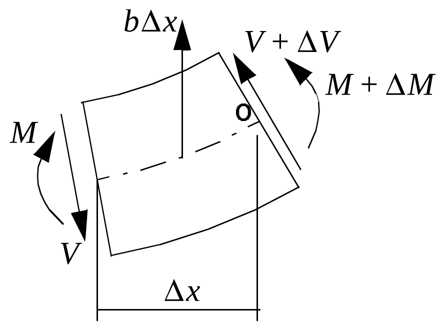
For optimal design $\alpha = 0.586$

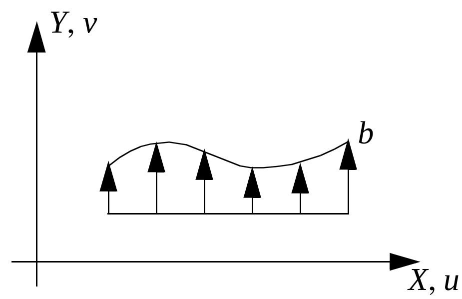
$$k_{opt} = \frac{48EI}{L^3}\frac{1}{1.067 - 1} = 716\frac{EI}{L^3}$$

3.3 Modeling of beam with multiple cables using a beam on elastic foundation model

A beam on many springs can be modelled as a beam on an elastic foundation. A simple analytic solution exists for "constant" foundation stiffness. This solution is useful for preliminary design.

3.3.1 Governing equations





For small rotations

$$\sum F_y = -V + V + \Delta V + b\Delta x = 0$$

$$\frac{\Delta V}{\Delta x} + b = 0$$

$$\frac{dV}{dx} + b = 0 \qquad \text{(i)}$$

and

$$\sum M_o = V\Delta x + M + \Delta M - M = 0$$

$$\frac{\Delta M}{\Delta x} + V = 0$$

$$\frac{dM}{dx} + V = 0 \qquad \text{(ii)}$$

From beam theory

$$\gamma = v,_x - \beta = \frac{V}{D_T}$$

$$\beta,_x = \frac{M}{D_B}$$

Neglecting transverse shear deformation (ie $\gamma = 0$)

$$v,_x = \beta$$

and

$$\beta,_x = v,_{xx}$$

Then

$$v_{,xx} = \frac{M}{D_B}$$

$$M = D_B v_{,xx}$$

From (ii)

$$-V = \frac{dM}{dx} = M_{,x}$$

$$-V_{,x} = \frac{d^2M}{dx^2} = M_{,xx}$$

replacing into (i)

$$-\frac{d^2M}{dx^2} + b = 0$$

$$-\frac{d^2}{dx^2}(D_B v_{,xx}) + b = 0$$

Boundary conditions

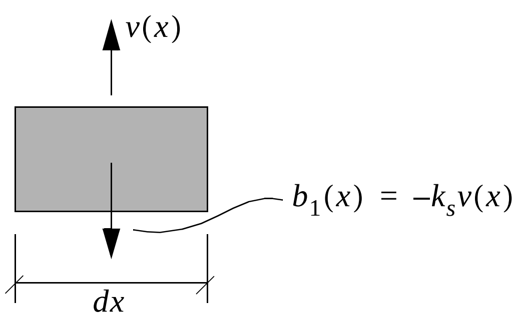
v or V prescribed at each end

and

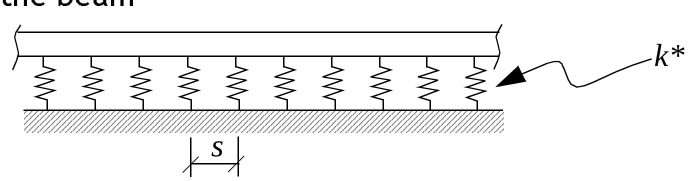
β or M prescribed at each end

3.3.2 Winkler Formulation Model

Winkler's hypothesis assumes the restraining force b at point x is in part a function of the displacement at x.



This relation is the limiting form when there are many closely-spaced uncoupled springs supporting the beam



For k^* and s constant

$$k_s = \frac{k^*}{s}$$

Then

$$b = -k_s v + \bar{b}$$

where $\bar{b}$ = some prescribed loading

In this case the governing equation takes the form

$$\frac{d^2}{dx^2}(D_B v,_{xx}) + k_s v = \bar{b}$$

Note: Boundary conditions have the same form and do not depend on the nature of the restraining foundation stiffness.

3.3.3 Solution with D_B and foundation stiffness constant

D_B = constant

k_s = constant

$$\frac{d^4 v}{dx^4} + \frac{k_s}{D_B} v = \frac{\bar{b}}{D_B}$$

Define

$$4\lambda^4 = \frac{k_s}{D_B}$$

Solution has the form

$$v = v_{part} + e^{-\lambda x}(C_1 \sin\lambda x + C_2 \cos\lambda x) + e^{\lambda x}(C_3 \sin\lambda x + C_4 \cos\lambda x)$$

where $C_1, C_2, C_3,$ and C_4 are integration constants.

<u>Particular solution</u>

$\bar{b}$ = constant

$$v_{part} = \frac{\bar{b}}{k_s}$$

<u>Characteristic length</u>

$e^{-\lambda x}$ decays with increasing x

for $\lambda x \geq 3 \;\; e^{-\lambda x} \approx 0$

For example

$$e^{-3} = 0.0495$$

$$e^{-4} = 0.0183$$

Define L_b as the "characteristic" length

$$L_b = \frac{3}{\lambda} = \frac{3}{\left\{\frac{k_s}{4D_B}\right\}^{1/4}} = 3\left\{\frac{4D_B}{k_s}\right\}^{1/4}$$

Then, for $x > L_b$, the $e^{-\lambda x}$ terms can be ignored.

Also, if $x > L_b$

$$v = v_{part}(x) + e^{\lambda x}(C_3 \sin\lambda x + C_4 \cos\lambda x)$$

Define

$$\sin\lambda x = \phi_1$$

$$\cos\lambda x = \phi_2$$

$$\cos\lambda x + \sin\lambda x = \phi_3$$

$$\cos\lambda x - \sin\lambda x = \phi_4$$

Then

$$\phi_1 C_3 + \phi_2 C_4 = \frac{v(x) - v_{part}(x)}{e^{\lambda x}}$$

C_3 and C_4 are evaluated using the B.C.s at $x = L$

$$\phi_1 C_3 + \phi_2 C_4 = \frac{v(L) - v_{part}(L)}{e^{\lambda L}} \qquad \text{(i)}$$

Also

$$v,_x = v,_{x(part)} + C_3\lambda(\cos\lambda x + \sin\lambda x) + C_4\lambda(\cos\lambda x - \sin\lambda x)$$

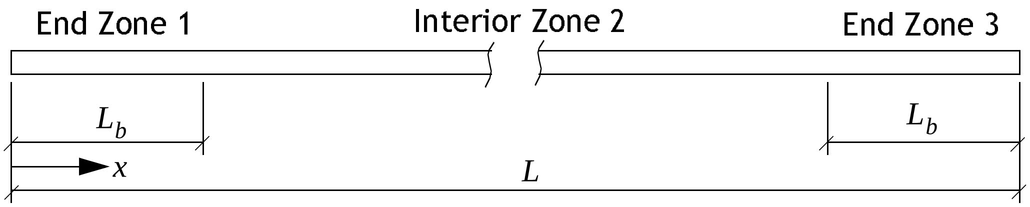
At $x = L$

$$\phi_1 \lambda C_3 + \phi_2 \lambda C_4 = \frac{v'(L) - v_{part}'(L)}{e^{\lambda L}} \qquad \text{(ii)}$$

From (i) and (ii) it can be deduced that C_3 and C_4 have a $e^{-\lambda L}$ factor. Therefore, the C_3 and C_4 terms can be neglected for $0 \le x \le L - L_b$

Therefore, the general solution can be approximated as

$$\begin{aligned} 0 \le x \le L_b \quad & v \approx v_p + e^{-\lambda x}(C_1 \sin\lambda x + C_2 \cos\lambda x) \\ L_b \le x \le L - L_b \quad & v \approx v_p \\ L - L_b \le x \le L \quad & v \approx v_p + e^{\lambda x}(C_3 \sin\lambda x + C_4 \cos\lambda x) \end{aligned}$$



The solution then consists of 2 end zone solutions and an interior zone solution when the member length is greater than $2L_b$

$$2L_b = 2\left(\frac{3}{\lambda}\right) = 6\left\{\frac{4D_B}{k_s}\right\}^{1/4}$$

3.3.4 Expansion of Solution Near $x = 0$

$$v \approx v_p + e^{-\lambda x}(C_1 \sin\lambda x + C_2 \cos\lambda x)$$

$$v,_x = \beta = v_p,_x + C_1\{\lambda e^{-\lambda x}(\cos\lambda x - \sin\lambda x)\} + C_2\{\lambda e^{-\lambda x}(-\cos\lambda x - \sin\lambda x)\}$$

$$v,_{xx} = \beta,_x = v_p,_{xx} + C_1\{\lambda^2 e^{-\lambda x}(-2\cos\lambda x)\} + C_2\{\lambda^2 e^{-\lambda x}(2\sin\lambda x)\}$$

$$M = D_B v,_{xx}$$

$$v,_{xxx} = v_p,_{xxx} + C_1\{2\lambda^3 e^{-\lambda x}(\cos\lambda x + \sin\lambda x)\} + C_2\{2\lambda^3 e^{-\lambda x}(\cos\lambda x - \sin\lambda x)\}$$

$$V = -D_B v,_{xxx}$$

Set

$$\psi_1 = e^{-\lambda x}(\cos\lambda x + \sin\lambda x)$$

$$\psi_2 = e^{-\lambda x}\sin\lambda x$$

$$\psi_3 = e^{-\lambda x}(\cos\lambda x - \sin\lambda x)$$

$$\psi_4 = e^{-\lambda x}\cos\lambda x$$

Then

$$v = v_p + C_1\psi_2 + C_2\psi_4$$

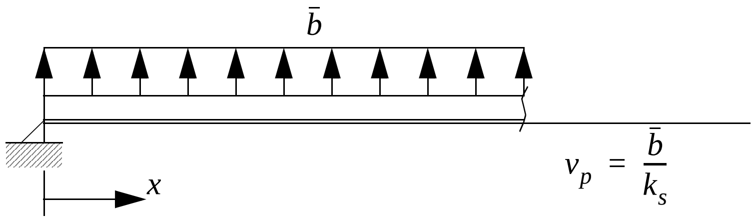
$$\beta = v_p,_x + C_1\lambda\psi_3 - C_2\psi_1$$

$$v,_{xx} = v_p,_{xx} + C_1\{-2\lambda^2\psi_4\} + C_2\{2\lambda^2\psi_2\}$$

$$v,_{xxx} = v_p,_{xxx} + C_1\{2\lambda^3\psi_1\} + C_2\{2\lambda^3\psi_3\}$$

3.3.5 Examples of Loadings and Boundary Conditions

Case 1



Boundary Conditions

At $x = 0$ $\quad v = 0$

$$M = 0 \rightarrow v,_{xx} = 0$$

$$\frac{\bar{b}}{k_s} + C_2 = 0 \rightarrow C_2 = -\frac{\bar{b}}{k_s}$$

$$2\lambda^2 C_1 = 0 \rightarrow C_1 = 0$$

Then

$$v = \frac{\bar{b}}{k_s}(1 - e^{-\lambda x}\cos\lambda x) = \frac{\bar{b}}{k_s}(1 - \psi_4)$$

$$\frac{M}{D_B} = \frac{\bar{b}}{k_s}(-2\lambda^2 e^{-\lambda x}\sin\lambda x) = -\frac{2\bar{b}}{k_s}\lambda^2\psi_2$$

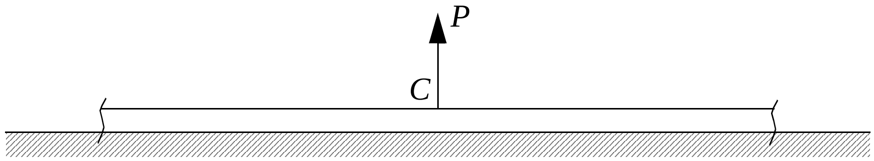
Maximum value of M_B at $\lambda x \cong 0.8 \rightarrow \psi_2\big|_{max} \cong 0.322$

$$\left.\frac{M}{D_B}\right|_{max} = -\frac{2\bar{b}}{k_s}\lambda^2(0.322) \text{ @ } x = \frac{0.8}{\lambda}$$

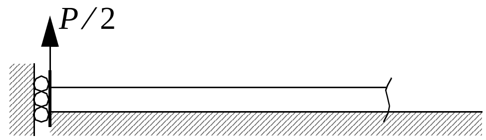
Maximum value of v at $\lambda x \cong 2.4 \rightarrow \psi_4\big|_{max} \cong -0.067$

$$v\big|_{max} = \frac{\bar{b}}{k_s}(1 + 0.067) = 1.067\frac{\bar{b}}{k_s} \text{ @ } x = \frac{2.4}{\lambda}$$

Case 2 - Infinitely long beam with concentrated force at center



Use symmetry at C to model as



Boundary Conditions

At $x = 0$ $\qquad V = -\frac{P}{2} \rightarrow D_B v,_{xxx} = \frac{P}{2}$

$$\beta = v,_x = 0$$

So

$$v,_x\big|_0 = C_1 - C_2 = 0 \rightarrow C_1 = C_2$$

and

$$v,_{xxx}\big|_0 = 2\lambda^3(C_1 + C_2) = \frac{P}{2D_B} \rightarrow C_1 = C_2 = \frac{P}{8D_B\lambda^3} = \frac{P\lambda}{2k_s}$$

The solution is

$$v = \frac{P\lambda}{2k_s}(\psi_2 + \psi_4)$$

$$\frac{M}{D_B} = -\frac{P\lambda^3}{k_s}\psi_3 \rightarrow M = \frac{P}{4\lambda}\psi_3$$

$$V = D_B(C_1\{2\lambda^3\psi_1\} + C_2\{2\lambda^3\psi_3\}) = -D_B\left(\frac{P\lambda}{2k_s}\right)(4\lambda^3\psi_4)$$

Maximum value of M at $x = 0$

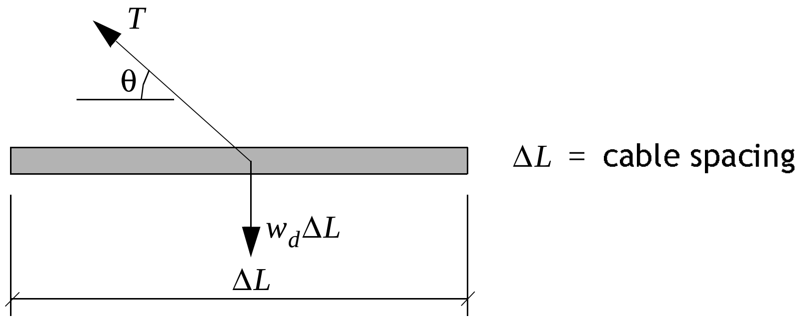
$$M\big|_{max} = -\frac{P\lambda^3}{k_s}D_B = -\frac{P}{4\lambda}$$

Maximum value of V at $x = 0$

$$V\big|_{max} = \frac{P\lambda}{2k_s}$$

3.4 Design procedures for cable/beam systems

3.4.1 Strength-Based Design



$$T \sin\theta = w_d \Delta L$$

Set σ_d = allowable stress for dead weight

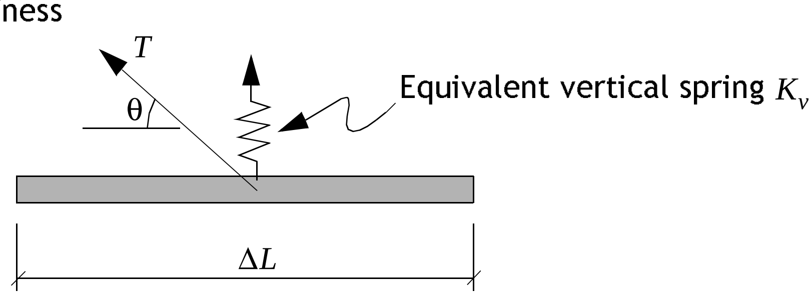
$$\sigma_d = f\sigma_u$$

$$f \cong 0.4 \text{ to } 0.5$$

Determine area of cable

$$A_c = \frac{1}{\sigma_d} T = \frac{w_d \Delta L}{\sigma_d \sin\theta}$$

Resulting Stiffness



k_c = stiffness of cable

$$K_v = k_c \sin^2\theta$$

$$k_c = \frac{AE^*}{L}$$

where

$$E^* = E_{eff} = E^*(T)$$

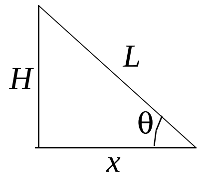
Then

$$K_v = \frac{AE^*}{L}\sin^2\theta = \frac{w_d \Delta L}{\sigma_d}\frac{E^*}{L}\sin\theta$$

Define $k_v^* = K_v / \Delta L$ = distributed stiffness

$$k_v^* = \frac{E^* w_d}{\sigma_d L}\sin\theta$$

Harp Cable Arrangement - θ = constant



$$x = L\cos\theta \rightarrow L = \frac{x}{\cos\theta}$$

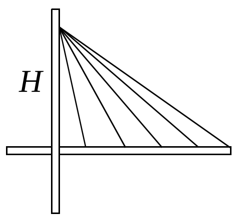
$$A_c = \frac{w_d \Delta L}{\sigma_d \sin\theta} = \text{constant}$$

$$k_v^*(x) = \frac{E^* w_d}{\sigma_d x}\cos\theta\sin\theta$$

Note

- Stiffness decreases rapidly with distance x from the tower
- k_v^* at $x = 0$ is ∞. Therefore, need to modify arrangement in this region

Fan Cable Arrangement - H = constant



$$L = \{H^2 + x^2\}^{1/2}$$

$$\sin\theta = \frac{H}{L} \qquad \cos\theta = \frac{x}{L} \qquad \tan\theta = \frac{H}{x}$$

$$k_v^*(x) = \frac{E^* w_d}{\sigma_d L}\frac{H}{L} = \frac{E^* w_d H}{\sigma_d L^2}$$

$$\boxed{k_v^*(x) = \frac{E^* w_d}{\sigma_d}\frac{H}{(H^2 + x^2)} = \frac{E^* w_d}{H\sigma_d}\frac{1}{1 + \left(\frac{x}{H}\right)^2}}$$

$$\boxed{A_c = \left\{\frac{w_d \Delta L}{\sigma_d}\right\}\left\{1 + \left(\frac{x}{H}\right)^2\right\}^{1/2}}$$

Tower Geometry

$$H \cong \alpha L_{max} = 2\alpha\frac{L_{max}}{2}$$

where $\qquad \alpha \cong \frac{1}{4}$

Then

$$\left(\frac{x}{H}\right)^2 \cong \frac{1}{4\alpha^2}\left\{\frac{x}{(L_{max}/2)}\right\}^2 \cong 4\left\{\frac{x}{(L_{max}/2)}\right\}$$

3.4.2 Stiffness-Based Design

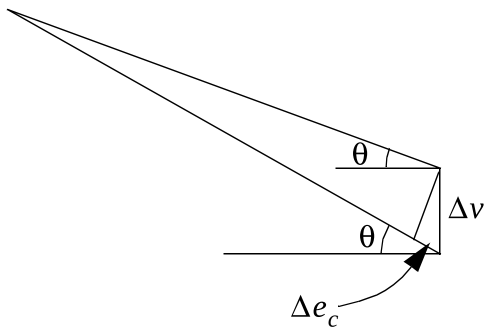
ΔL = cable spacing

k_v* = constant

$$K*_v = \Delta L k*_v = \frac{AE}{L_c}\sin^2\theta$$

$$A_c = \frac{(k*_v \Delta L)L_c}{E*\sin^2\theta}$$

Check for strength



Let Δv = movement due to live load and

ΔT_c = Increment in cable tension due to live load

$$\Delta T_c = k_c \Delta e_c$$

$$\Delta e_c = \Delta v \sin\theta$$

$$\Delta T_c = \left(\frac{A_c E*_c}{L_c}\sin\theta\right)\Delta v = \left(\frac{k*_v \Delta L}{\sin\theta}\right)\Delta v$$

T_T = total tension in cable $= T_c + \Delta T_c$

$$T_c = \frac{w_d}{\alpha}\frac{\Delta L}{\sin\theta}$$

$$T_T = \frac{\Delta L}{\sin\theta}\left\{\frac{w_d}{\alpha} + k*_v \Delta v\right\} = \frac{\Delta L}{\sin\theta}\{w_{dead} + w_{live}\}$$

Harp Cable Arrangement - θ = constant

$$L_c = \frac{x}{\cos\theta}$$

$$\boxed{A_c = \frac{(k*_v \Delta L)}{E*}\frac{x}{\cos\theta\sin^2\theta}}$$

Fan Cable Arrangement

$H = L_c \sin\theta$ = constant

$$A_c = \frac{(k*_v \Delta L)}{E*}\frac{L_c^3}{H^2} = \frac{(k*_v \Delta L)}{E*}\frac{(H^2 + x^2)^{3/2}}{H^2}$$

$$\boxed{A_c = \frac{(k*_v \Delta L)}{E*}H\left\{1 + \left(\frac{x}{H}\right)^2\right\}^{3/2}}$$