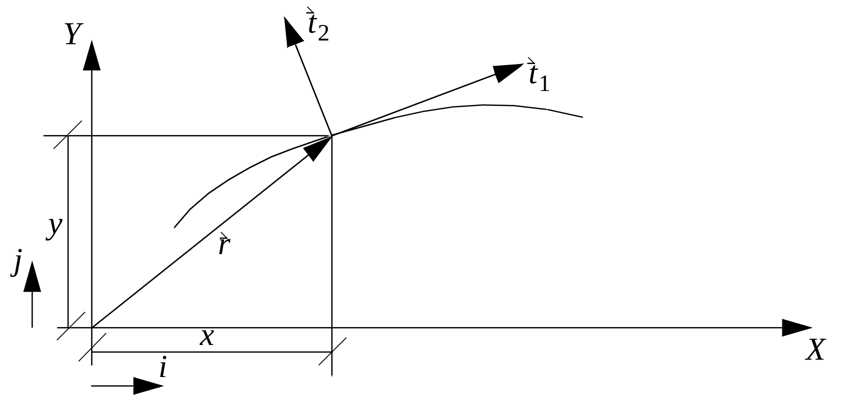
2.1 Geometric Relations: Plane Curve



$$x = x(\theta)$$
$$y = y(\theta)$$
θ = Parameter to define the curve
$\vec{r}$ = position vector for a point on the curve
$$\vec{r} = x\vec{i} + y\vec{j}$$

2.1.1 Arc length

Differential arc length = $(dx^2 + dy^2)^{1/2} = ds$

$$ds = \left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right)^{1/2} d\theta$$

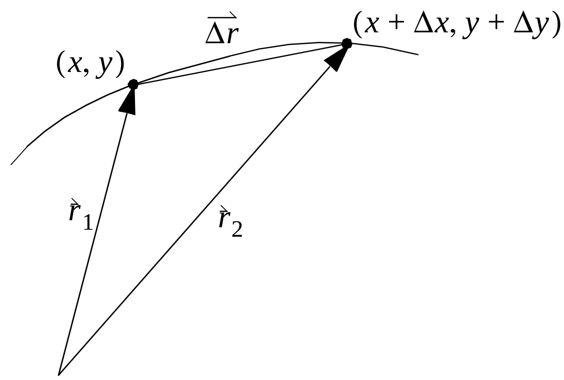
Define

$$\alpha = \left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right)^{1/2}$$

Then

$$ds = \alpha d\theta$$

2.1.2 Tangent and Normal Vectors



$$\overrightarrow{\Delta r} = \vec{r}_2 - \vec{r}_1 = \vec{r}(\theta + \Delta\theta) - \vec{r}(\theta)$$

$$\overrightarrow{\Delta r} = ((x + \Delta x)\vec{i} + (y + \Delta y)\vec{j}) - (x\vec{i} + y\vec{j})$$

$$\overrightarrow{\Delta r} = (\Delta x\vec{i} + \Delta y\vec{j})$$

$$|\overrightarrow{\Delta r}| = (\Delta x^2 + \Delta y^2)^{1/2} = \Delta s$$

Then for continuous and well-behaved functions

$$d\vec{r} = (dx\vec{i} + dy\vec{j})$$

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{|d\vec{r}|} = \vec{t}_1 = \left(\frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j}\right)$$

$\vec{t}_1$ = unit tangent vector

$$\vec{t}_1 = \left(\frac{1}{\alpha}\frac{dx}{d\theta}\right)\vec{i} + \left(\frac{1}{\alpha}\frac{dy}{d\theta}\right)\vec{j}$$

Define $\vec{t}_2$ such that

$$\vec{t}_1 \times \vec{t}_2 = \vec{k} \text{ (unit vector in } z\text{-direction)}$$

and enforce

$$\vec{t}_1 \bullet \vec{t}_2 = 0 \text{ (perpendicular)}$$

gives

$$\vec{t}_2 = \left(-\frac{1}{\alpha}\frac{dy}{d\theta}\right)\vec{i} + \left(\frac{1}{\alpha}\frac{dx}{d\theta}\right)\vec{j}$$

2.1.3 Differentiation formulae for the tangent and normal vectors $\vec{t}_1$ and $\vec{t}_2$

$$\vec{t}_1 \bullet \vec{t}_1 = |\vec{t}_1|^2 = 1$$

$$\vec{t}_2 \bullet \vec{t}_2 = |\vec{t}_2|^2 = 1$$

$$\frac{d}{ds}(\vec{t}_1 \bullet \vec{t}_1) = \vec{t}_1 \bullet \frac{d\vec{t}_1}{ds} + \vec{t}_1 \bullet \frac{d\vec{t}_1}{ds} = 2\left(\vec{t}_1 \bullet \frac{d\vec{t}_1}{ds}\right) = 0$$

Therefore, $\frac{d\vec{t}_1}{ds}$ must be $\perp$ to $\vec{t}_1$, and thus proportional to $\vec{t}_2$

The result is written as

$$\frac{d\vec{t}_1}{ds} = \frac{1}{R}\vec{t}_2$$

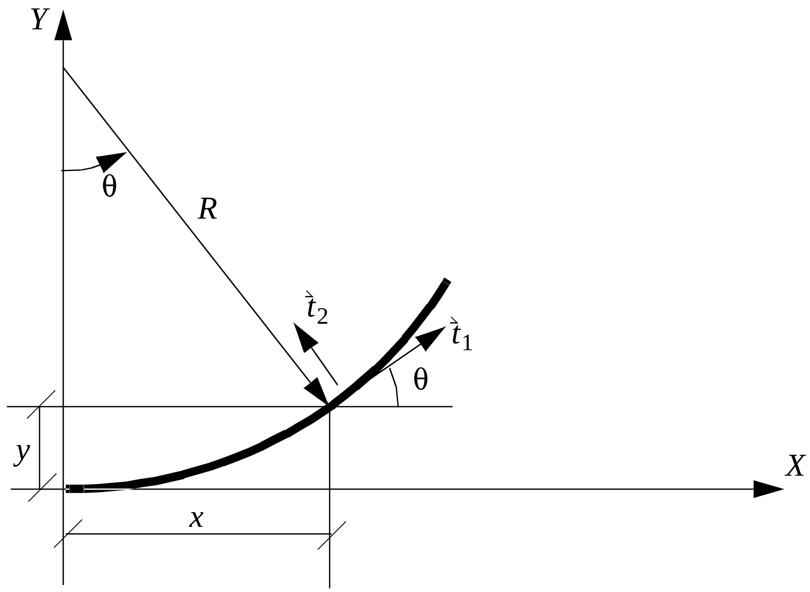
$$\frac{d\vec{t}_2}{ds} = -\frac{1}{R}\vec{t}_1$$

where

$$\frac{1}{R} = \frac{1}{\alpha^3}\left(-\frac{d^2x}{d\theta^2}\frac{dy}{d\theta} + \frac{d^2y}{d\theta^2}\frac{dx}{d\theta}\right)$$

2.1.4 Example: Curvilinear Formulation

Take the angle θ as the parameter defining the curve



$$x = R\sin\theta$$

$$y = R(1 - \cos\theta)$$

$$\alpha = R\left\{\left(\frac{d}{d\theta}\sin\theta\right)^2 + \left(\frac{d}{d\theta}(1 - \cos\theta)\right)^2\right\}^{1/2}$$

$$\alpha = R\{\cos^2\theta + \sin^2\theta\}^{1/2} = R$$

$$\vec{t}_1 = \frac{1}{R}\{R\cos\theta\vec{i} + R\sin\theta\vec{j}\} = \cos\theta\vec{i} + \sin\theta\vec{j}$$

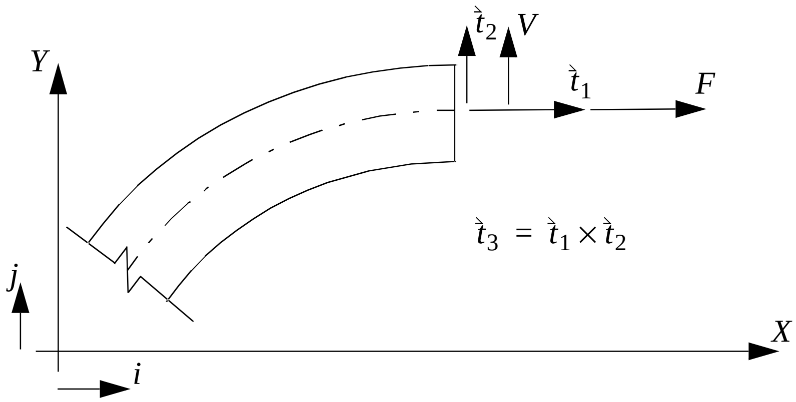
$$\vec{t}_2 = -\sin\theta\vec{i} + \cos\theta\vec{j}$$

$$\frac{d\vec{t}_1}{ds} = \frac{1}{\alpha}\frac{d\vec{t}_1}{ds} = \frac{1}{R}\frac{d\vec{t}_1}{ds} = \frac{1}{R}(-\sin\theta\vec{i} + \cos\theta\vec{j}) = \frac{1}{R}\vec{t}_2$$

$$\frac{d\vec{t}_2}{ds} = \frac{1}{\alpha}\frac{d\vec{t}_2}{ds} = \frac{1}{R}\frac{d\vec{t}_2}{ds} = \frac{1}{R}(-\cos\theta\vec{i} - \sin\theta\vec{j}) = -\frac{1}{R}\vec{t}_1$$

This illustrates the equations on the previous page.

2.2 Force Equilibrium Formulation: Curvilinear Formulation



$$\vec{F} = F\vec{t}_1 + V\vec{t}_2$$

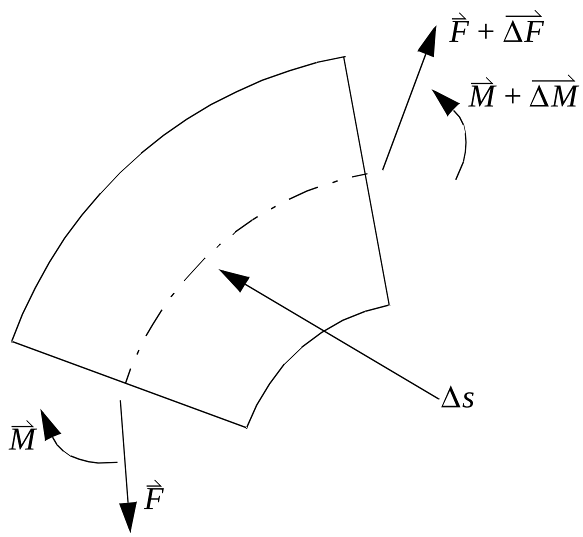
$$\vec{M} = M\vec{t}_3$$

Loading per unit arc length

$$\vec{b} = b_1\vec{t}_1 + b_2\vec{t}_2$$

$$\vec{m} = m\vec{t}_3$$

2.2.1 Vector Equations



$$\vec{F} + \Delta\vec{F} - \vec{F} + \vec{b}\Delta s = 0$$

$$\vec{M} + \Delta\vec{M} - \vec{M} + \vec{m}\Delta s - \Delta s\vec{t}_1 \times (-\vec{F}) = 0$$

Then

$\dfrac{d\vec{F}}{ds} + \vec{b} = 0$
$\dfrac{d\vec{M}}{ds} + \vec{m} + \vec{t}_1 \times \vec{F} = 0$

2.2.2 Scalar Equations

$$\vec{F} = F\vec{t}_1 + V\vec{t}_2$$

$$\frac{d\vec{F}}{ds} = \frac{d}{ds}(F\vec{t}_1 + V\vec{t}_2) = F\frac{d\vec{t}_1}{ds} + \vec{t}_1\frac{dF}{ds} + V\frac{d\vec{t}_2}{ds} + \vec{t}_2\frac{dV}{ds}$$

$$\frac{d\vec{F}}{ds} = \frac{F}{R}\vec{t}_2 + \vec{t}_1\frac{dF}{ds} - \frac{V}{R}\vec{t}_1 + \vec{t}_2\frac{dV}{ds}$$

$$\frac{d\vec{F}}{ds} + \vec{b} = \frac{F}{R}\vec{t}_2 + \vec{t}_1\frac{dF}{ds} - \frac{V}{R}\vec{t}_1 + \vec{t}_2\frac{dV}{ds} + b_1\vec{t}_1 + b_2\vec{t}_2$$

Separating into components of $\vec{t}_1$ and $\vec{t}_2$ leads to

$$\frac{dF}{ds} - \frac{V}{R} + b_1 = 0$$

$$\frac{dV}{ds} + \frac{F}{R} + b_2 = 0$$

For the moment equation

$$\frac{d\vec{M}}{ds} + \vec{m} + \vec{t}_1 \times (F\vec{t}_1 + V\vec{t}_2) = 0$$

Expanding and writing in scalar form

$$\frac{dM}{ds} + m + V = 0$$

Summary

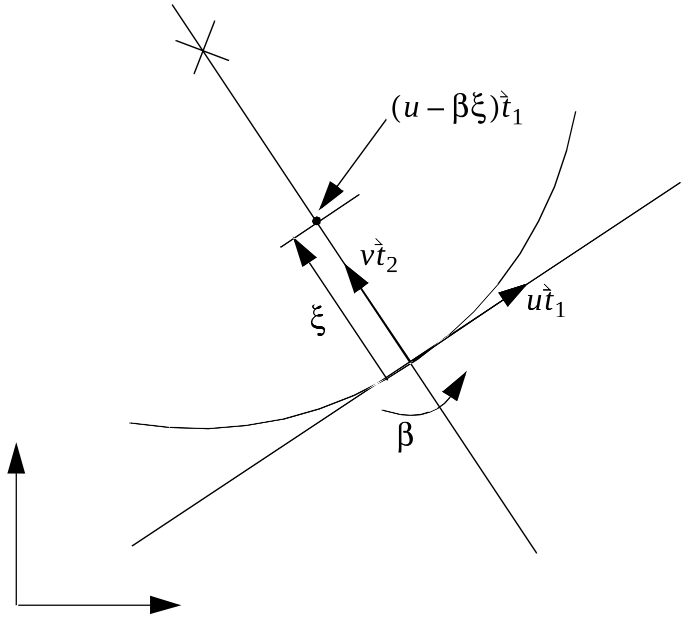
$$\frac{dF}{ds} - \frac{V}{R} + b_1 = 0$$

$$\frac{dV}{ds} + \frac{F}{R} + b_2 = 0$$

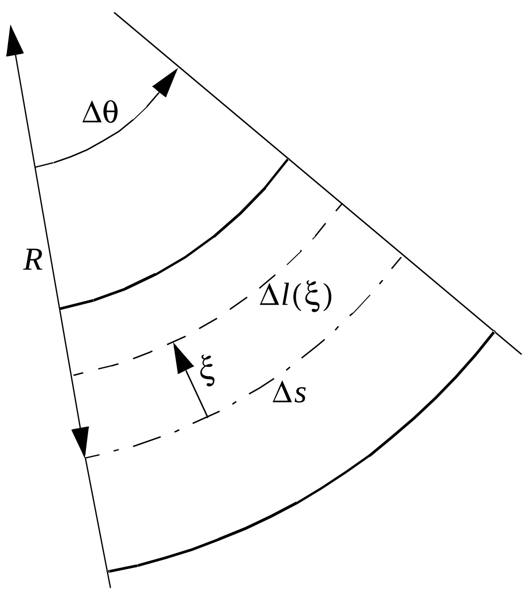
$$\frac{dM}{ds} + m + V = 0$$

Note: F and V are coupled

2.3 Force Displacement Relations : Curvilinear Coordinates



2.3.1 Strains : Displacement components defined wrt the local frame $(\vec{t}_1, \vec{t}_2)$



$$\Delta l(\xi) = (R - \xi)\Delta\theta$$

$$\Delta s = R\Delta\theta \rightarrow \Delta\theta = \frac{\Delta s}{R}$$

$$\Delta l(\xi) = (R - \xi)\frac{\Delta s}{R}$$

$$\Delta l(\xi) = \left(1 - \frac{\xi}{R}\right)\Delta s$$

$$\vec{u} = u\vec{t}_1 + v\vec{t}_2 - \xi\beta\vec{t}_1$$

$$\vec{\varepsilon} = \begin{bmatrix} \varepsilon \\ \gamma \end{bmatrix}$$

$$\vec{\varepsilon} = \frac{d\vec{u}}{dl}$$

$$dl = \left(1 - \frac{\xi}{R}\right) ds$$

$$\vec{\varepsilon} = \frac{1}{\left(1 - \frac{\xi}{R}\right)} \frac{d\vec{u}}{ds}$$

$$\frac{d\vec{u}}{ds} = \frac{u}{R}\vec{t}_2 + \frac{du}{ds}\vec{t}_1 - \frac{v}{R}\vec{t}_1 + \frac{dv}{ds}\vec{t}_2 - \xi\frac{\beta}{R}\vec{t}_2 - \xi\frac{d\beta}{ds}\vec{t}_1$$

Separating $\vec{t}_1$ and $\vec{t}_2$ terms

$$\varepsilon = \frac{1}{\left(1 - \frac{\xi}{R}\right)} \left\{ \frac{du}{ds} - \frac{v}{R} - \xi\frac{d\beta}{ds} \right\}$$

Define

$$\varepsilon_a = \frac{du}{ds} - \frac{v}{R}$$

$$\varepsilon = \frac{1}{\left(1 - \frac{\xi}{R}\right)} \left\{ \varepsilon_a - \xi\frac{d\beta}{ds} \right\}$$

$$\gamma = \frac{1}{\left(1 - \frac{\xi}{R}\right)} \left\{ \frac{dv}{ds} - \frac{\xi}{R}\beta + \frac{u}{R} - \beta \right\}$$

$$\gamma = \frac{1}{\left(1 - \frac{\xi}{R}\right)} \gamma_o$$

For most members $\xi \ll R$ then

$$\frac{\xi}{R} \sim 0$$

so

$$\gamma_o \cong \frac{dv}{ds} - \beta + \frac{u}{R}$$

Also if the member is thin

$$1 - \frac{\xi}{R} \sim 1$$

Then, the strain distribution reduces to a linear distribution over the cross-section.

2.3.2 Stress Strain Relations

$$\varepsilon = \frac{1}{E}\sigma + \varepsilon_o \rightarrow \sigma = E(\varepsilon - \varepsilon_o)$$

$$\gamma = \frac{1}{G}\tau \rightarrow \tau = G\gamma$$

2.3.3 Force-Deformation Relations

$$F = \int \sigma dA$$

$$M = \int -\xi \sigma dA$$

$$V = \int \tau dA$$

Substituting for the strains leads to

$$F = \int E(\varepsilon - \varepsilon_o) dA = \int \left\{ \frac{E}{1 - \frac{\xi}{R}} \left(\varepsilon_a - \xi \frac{d\beta}{ds} \right) - E\varepsilon_o \right\} dA$$

$$F = \int \frac{E}{\left(1 - \frac{\xi}{R}\right)} \frac{du}{ds} dA - \int \frac{E}{\left(1 - \frac{\xi}{R}\right)} \frac{v}{R} dA - \int \frac{E}{\left(1 - \frac{\xi}{R}\right)} \xi \frac{d\beta}{ds} dA - \int E\varepsilon_o dA$$

$$F = \varepsilon_a \int \frac{E}{\left(1 - \frac{\xi}{R}\right)} dA - \beta,_s \int \frac{E\xi}{\left(1 - \frac{\xi}{R}\right)} dA - \int E\varepsilon_o dA$$

Similarly

$$M = -\varepsilon_a \int \frac{E\xi}{\left(1 - \frac{\xi}{R}\right)} dA + \beta,_s \int \frac{E\xi^2}{\left(1 - \frac{\xi}{R}\right)} dA + \int \xi E\varepsilon_o dA$$

$$V = \gamma_o \int \frac{G}{\left(1 - \frac{\xi}{R}\right)} dA$$

Geometric Relations

$$\frac{1}{1-\frac{\xi}{R}} = \frac{R}{R-\xi} = 1+\frac{\xi}{R-\xi} = 1+\frac{1}{R}\frac{\xi}{1-\frac{\xi}{R}}$$

$$\int_A \frac{1}{1-\frac{\xi}{R}}dA = \int_A dA + \frac{1}{R}\int_A \frac{\xi}{1-\frac{\xi}{R}}dA$$

Now

$$\frac{\xi}{1-\frac{\xi}{R}} = \xi\frac{1}{1-\frac{\xi}{R}} = \xi+\frac{1}{R}\frac{\xi^2}{1-\frac{\xi}{R}}$$

$$\frac{1}{R}\int_A \frac{\xi}{1-\frac{\xi}{R}}dA = \int_A \frac{\xi}{R}dA + \frac{1}{R^2}\int_A \frac{\xi^2}{1-\frac{\xi}{R}}dA$$

Making

$$\int_A \frac{1}{1-\frac{\xi}{R}}dA = \int_A dA + \int_A \frac{\xi}{R}dA + \frac{1}{R^2}\int_A \frac{\xi^2}{1-\frac{\xi}{R}}dA$$

But

$\int_A \xi dA = 0$ if ξ is measured from the centroid of the section

Finally, one can write

$$\int_A \frac{1}{1-\frac{\xi}{R}}dA = \int_A dA + \frac{1}{R^2}\int_A \frac{\xi^2}{1-\frac{\xi}{R}}dA = A+\frac{I'}{R^2}$$

where

$$I' = \int_A \frac{\xi^2}{1-\frac{\xi}{R}}dA$$

We can use this expression to expand the force-deformation relations.

Summarizing

$$
\boxed{\begin{aligned}
F &= D_S\varepsilon_a + D_{SB}\beta,_s + F_o \\
M &= D_{SB}\varepsilon_a + D_B\beta,_s + M_o \\
V &= D_T\gamma_o
\end{aligned}}
$$

where

$$D_S = \int \frac{E}{\left(1 - \frac{\xi}{R}\right)} dA$$

$$D_{SB} = -\int \frac{\xi E}{\left(1 - \frac{\xi}{R}\right)} dA$$

$$D_T = \int \frac{G}{\left(1 - \frac{\xi}{R}\right)} dA$$

$$D_B = \int \frac{\xi^2 E}{\left(1 - \frac{\xi}{R}\right)} dA$$

For the homogeneous case E and G are constant, and the above simplify to

$$D_S = EA + E\frac{I'}{R^2}$$

$$D_{SB} = -E\frac{I'}{R}$$

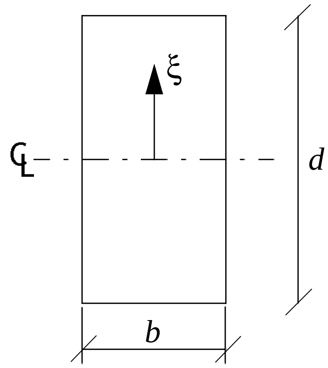
$$D_B = EI'$$

$$D_T = GA + G\frac{I'}{R^2}$$

Additionally, for a thin curved member, one can neglect ξ/R wrt 1, obtaining the simplified form

$$
\boxed{\begin{aligned}
D_S &= EA + \frac{EI}{R^2} \\
D_{SB} &= -\frac{EI}{R} \\
D_B &= EI \\
D_T &= GA + \frac{GI}{R^2}
\end{aligned}}
$$

2.3.4 Example for a Rectangular Section



$$I' = \int_A \frac{\xi^2}{1 - \frac{\xi}{R}} dA$$

$$dA = b d\xi$$

$$I' = \int_{-d/2}^{d/2} \frac{b\xi^2}{1 - \frac{\xi}{R}} d\xi$$

$$I' = I\left\{1 + \frac{3}{20}\left(\frac{d}{R}\right)^2 + \frac{3}{112}\left(\frac{d}{R}\right)^4 + \ldots\right\}$$

$$I = \frac{bd^3}{12}$$

and for $\frac{d}{R} < 1$, $\left(\frac{d}{R}\right)^n \ll 1$ for $n > 1$

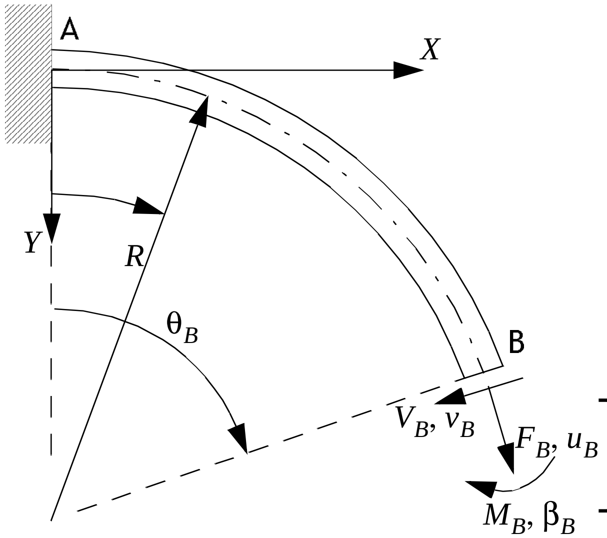
Summary of Equations

$$F = D_S\left(\frac{du}{ds} - \frac{v}{R}\right) + D_{SB}\frac{d\beta}{ds} + F_o$$

$$M = D_{SB}\left(\frac{du}{ds} - \frac{v}{R}\right) + D_B\frac{d\beta}{ds} + M_o$$

$$V = D_T\left(\frac{dv}{ds} - \beta + \frac{u}{R}\right)$$

2.3.5 Flexibility Matrix - Circular Member



Note: $\bar{k}$ into page

Assumptions

1. "Thin" linear elastic member
2. Neglect stretching and transvers shear deformations. This is valid for non-sha llow cases only

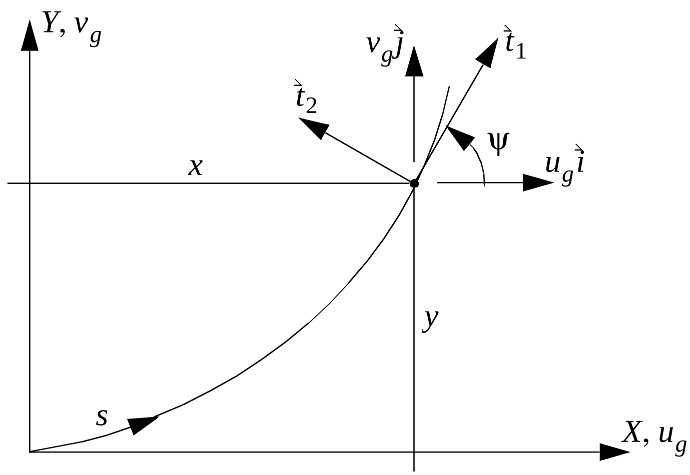
$$\underline{u}_B = \begin{bmatrix} u_B \\ v_B \\ \beta_B \end{bmatrix} \qquad \underline{F}_B = \begin{bmatrix} \bar{F}_B \\ \bar{V}_B \\ \bar{M}_B \end{bmatrix}$$

$$\underline{u}_B = \underline{f}_B \underline{F}_B$$

$$\underline{f}_B = \begin{bmatrix} R^2\left(\frac{3}{2} - 2\sin\theta_B + \frac{1}{2}\sin\theta_B\cos\theta_B\right) & R^2\left(1 - \cos\theta_B - \frac{1}{2}\sin^2\theta_B\right) & R(\theta_B - \sin\theta_B) \\ R^2\left(1 - \cos\theta_B - \frac{1}{2}\sin^2\theta_B\right) & R^2(\theta_B - \sin\theta_B\cos\theta_B) & R(1 - \cos\theta_B) \\ R(\theta_B - \sin\theta_B) & R(1 - \cos\theta_B) & \theta_B \end{bmatrix}$$

2.4 Cartesian Formulation - Governing Equations

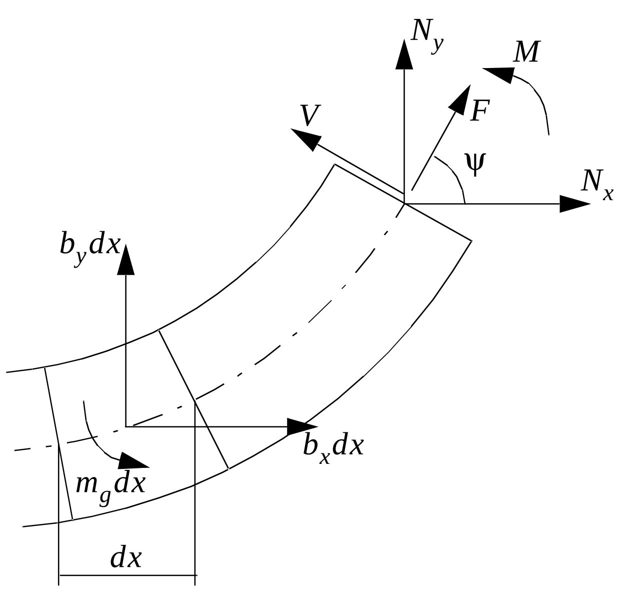
Consider the case where the centroidal axis is defined by $y = y(x)$. The independent variable is x, the position coordinate on the x-axis. In the angular formulation, we worked with displacement components and loads referred to the local curvilinear frame defined by the unit vectors $\vec{t}_1$ and $\vec{t}_2$. In this section, we are going to work with quantities referred to the global cartesian reference frame. This approach was originally suggested by Marguerre.



$$\alpha = \frac{ds}{dx} \rightarrow ds = \alpha dx$$

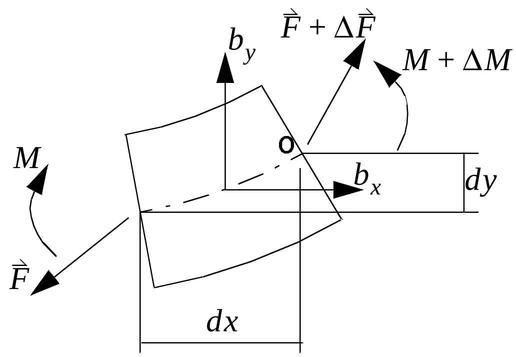
$$\sin\psi = \frac{dy}{ds} = \frac{dy}{\alpha dx}$$

$$\cos\psi = \frac{dx}{ds} = \frac{1}{\alpha}$$



The above figure defines the "cartesian" notation. The distributed loading (b_x, b_y, m_g) is defined with respect the the x and y directions and the loading per unit projected length (dx). Similarly, the displacement components (u_g, v_g) are referred to the cartesian global reference directions.

2.4.1 Force Equilibrium Equations



$$\sum F = 0 = \vec{F} + \Delta\vec{F} + \Delta x(b_x\vec{i} + b_y\vec{j}) - \vec{F} = 0$$

$$\frac{\Delta\vec{F}}{\Delta x} + b_x\vec{i} + b_y\vec{j} = 0$$

then

$$\frac{d\vec{F}}{dx} + b_x\vec{i} + b_y\vec{j} = 0$$

Knowing

$$\vec{F} = (F\cos\psi - V\sin\psi)\vec{i} + (F\sin\psi + V\cos\psi)\vec{j}$$

we get

$$\frac{d}{dx}((F\cos\psi - V\sin\psi)\vec{i} + (F\sin\psi + V\cos\psi)\vec{j}) + b_x\vec{i} + b_y\vec{j} = 0$$

Separating

$$\frac{d}{dx}(F\cos\psi - V\sin\psi) + b_x = 0$$

$$\frac{d}{dx}(F\sin\psi + V\cos\psi) + b_y = 0$$

Similarly

$$\sum M_o = 0 = -(\Delta x\vec{i} + \Delta y\vec{j}) \times (-\vec{F}) + (M + \Delta M - M + m_g\Delta x) - \frac{1}{2}(\Delta x\vec{i} + \Delta y\vec{j}) \times (b_x\vec{i} + b_y\vec{j})\Delta x$$

Second order term goes to 0 in the limit $\Delta x \to 0$

$$(\Delta x\vec{i} + \Delta y\vec{j}) \times ((F\cos\psi - V\sin\psi)\vec{i} + (F\sin\psi + V\cos\psi)\vec{j}) + (\Delta M + m_g\Delta x)\vec{k} = 0$$

$$(\Delta x(F\sin\psi + V\cos\psi) - \Delta y(F\cos\psi - V\sin\psi) + \Delta M + m_g\Delta x)\vec{k} = 0$$

$$F\sin\psi + V\cos\psi - \frac{\Delta y}{\Delta x}F\cos\psi - \frac{\Delta y}{\Delta x}V\sin\psi + \frac{\Delta M}{\Delta x} + m_g = 0$$

$$F\sin\psi + V\cos\psi - \frac{dy}{dx}F\cos\psi - \frac{dy}{dx}V\sin\psi + \frac{dM}{dx} + m_g = 0$$

But

$$\frac{dy}{dx} = \alpha\sin\psi \qquad \cos\psi = \frac{1}{\alpha}$$

Then

$$F\sin\psi + V\cos\psi - \alpha\sin\psi F\frac{1}{\alpha} - \alpha\sin\psi V\sin\psi + \frac{dM}{dx} + m_g = 0$$

$$V\cos\psi + \alpha\sin^2\psi V + \frac{dM}{dx} + m_g = 0$$

Multiply both sides by $\frac{1}{\alpha}$

$$\frac{1}{\alpha}\left(V\cos\psi + \alpha\sin^2\psi V + \frac{dM}{dx} + m_g\right) = 0$$

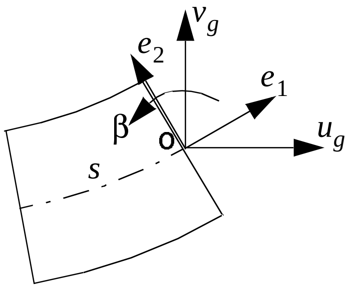
$$V\cos^2\psi + V\sin^2\psi + \frac{1}{\alpha}\frac{dM}{dx} + \frac{1}{\alpha}m_g = 0$$

Therefore

$$\boxed{\begin{aligned} &\frac{1}{\alpha}\frac{dM}{dx} + V + \frac{1}{\alpha}m_g = 0 \\ &\frac{d}{dx}(F\cos\psi - V\sin\psi) + b_x = 0 \\ &\frac{d}{dx}(F\sin\psi + V\cos\psi) + b_y = 0 \end{aligned}}$$

$$\boxed{\begin{aligned} N_x &= F\cos\psi - V\sin\psi \\ N_y &= F\sin\psi + V\cos\psi \end{aligned}}$$

4.2 Deformation - Displacement Relations



e_1 = stretching deformation

e_2 = transverse shear deformation

$$\frac{dx}{ds} = \cos\psi \rightarrow ds = \frac{dx}{\cos\psi}$$

$$\vec{u} = (u_g\vec{i} + v_g\vec{j})$$

$$\frac{d\vec{u}}{ds} = \frac{d\vec{u}}{dx}\cos\psi$$

$$\frac{d\vec{u}}{dx} = \left(\frac{du_g}{dx}\vec{i} + \frac{dv_g}{dx}\vec{j}\right)$$

$$\frac{d\vec{u}}{ds} = \cos\psi\left(\frac{du_g}{dx}\vec{i} + \frac{dv_g}{dx}\vec{j}\right)$$

$$e_1 = \frac{d\vec{u}}{ds} \cdot \vec{t}_1 = \cos\psi\left(\frac{du_g}{dx}\vec{i} + \frac{dv_g}{dx}\vec{j}\right) \cdot (\cos\psi\vec{i} + \sin\psi\vec{j})$$

$$\boxed{e_1 = \cos^2\psi\frac{du_g}{dx} + \sin\psi\cos\psi\frac{dv_g}{dx}}$$

$$e_2 = \frac{d\vec{u}}{ds} \cdot \vec{t}_2 - \beta = \cos\psi\left(\frac{du_g}{dx}\vec{i} + \frac{dv_g}{dx}\vec{j}\right) \cdot (-\sin\psi\vec{i} + \cos\psi\vec{j}) - \beta$$

$$\boxed{e_2 = -\sin\psi\cos\psi\frac{du_g}{dx} + \cos^2\psi\frac{dv_g}{dx} - \beta}$$

$$\boxed{k = \text{bending deformation} = \frac{d\beta}{ds} = \cos\psi\frac{d\beta}{dx}}$$

note: e_1 and e_2 are referred to the centroidal axes (s)

2.4.3 Force Deformation Relations

2.4.3.1 Thin slightly curved member: use relations developed for the prismatic case

Assume

$$\varepsilon = e_1$$

$$\gamma = e_2$$

$$F = \int_A E(e_1 - e_{1,o})dA = (e_1 - e_{1,o})D_S$$

$$D_S = \int_A EdA$$

$$V = \int_A Ge_2 dA = e_2 D_T$$

$$D_T = \int_A GdA$$

$$M = \int_A E(e_1 - e_{1,o})\xi dA = \int_A E(k - k_o)\xi^2 dA = (k - k_o)\int_A E\xi^2 dA$$

$$M = (k - k_0)D_B$$

$$D_B = \int_A E\xi^2 dA$$

Summarize as follows

$$e_1 = e_{1,o} + \frac{F}{D_S}$$

$$e_2 = \frac{V}{D_T}$$

$$k = k_o + \frac{M}{D_B}$$

2.4.3.2 Thick Case ($\xi \neq 0$ wrt R): change in l_o as a function of ξ must be considered

Use equations developed in section 2.3.3 with

$$e_a = e_1$$

$$\gamma_o = e_2$$

$$k = \frac{d\beta}{ds}$$

4.4 Approximate Formulation - Shallow Cartesian

- Shallow assumption $\psi^2 \ll 1$

Then $\cos\psi \cong 1$

$$\sin\psi \cong \tan\psi \cong \frac{dy}{dx} = y,_x$$

$$\alpha = 1 \qquad (ds \cong dx)$$

- Marguerre neglects V terms in the x equilibrium equation

$$\boxed{\begin{aligned} N_x &= F \\ N_y &= V + Fy,_x \end{aligned}}$$

So, resulting equations (shallow and Marguerre)

Equilibrium

$$\frac{dF}{dx} + b_x = 0$$

$$\frac{dV}{dx} + \frac{d}{dx}\left(F\frac{dy}{dx}\right) + b_y = 0$$

$$\frac{dM}{dx} + V + m = 0$$

Deformation - Displacement

$$e_1 = \frac{du_g}{dx} + \frac{dy}{dx}\frac{dv_g}{dx} = \varepsilon_a$$

$$e_2 = \frac{dv_g}{dx} - \beta = \gamma_o$$

$$k = \frac{d\beta}{dx}$$

For the Force-Displacement Relations, thin and thick should be considered separately

- Thin, Shallow, Marquerre

$$F = D_S\left(\frac{du_g}{dx} + \frac{dy}{dx}\frac{dv_g}{dx}\right) - D_S e,_o$$

$$D_S = \int_A E dA$$

$$V = D_T\left(\frac{dv_g}{dx} - \beta\right)$$

$$D_T = \int_A G dA$$

$$M = D_B\frac{d\beta}{dx} - D_B k_o$$

$$D_B = \int_A E\xi^2 dA$$

- Thick, Shallow, Marguerre

$$\left.\begin{aligned} F &= D_S\left(\frac{du_g}{dx} + \frac{dy}{dx}\frac{dv_g}{dx}\right) + D_{SB}\frac{d\beta}{ds} \\ M &= D_{SB}\left(\frac{du_g}{dx} + \frac{dy}{dx}\frac{dv_g}{dx}\right) + D_B\frac{d\beta}{ds} \end{aligned}\right\} ds \cong dx$$

$$V = D_T\left(\frac{dv_g}{dx} - \beta\right)$$

$$D_S = \int\frac{E}{(1-\xi/R)}dA$$

$$D_{SB} = -\int\frac{\xi E}{(1-\xi/R)}dA$$

$$D_T = \int\frac{G}{(1-\xi/R)}dA$$

$$D_B = \int\frac{\xi^2 E}{(1-\xi/R)}dA$$

Boundary Conditions

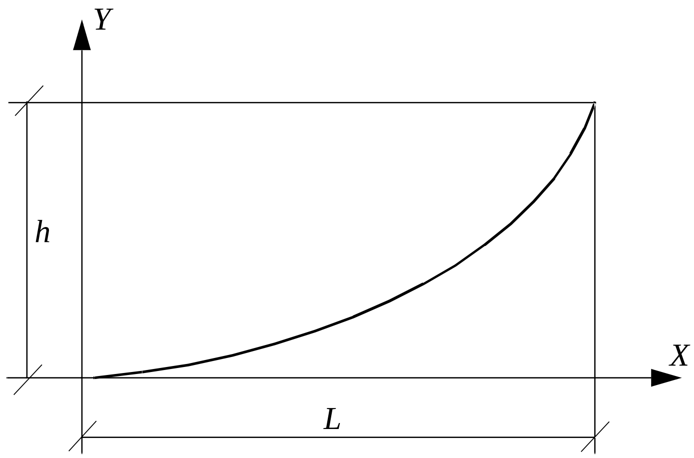
in general, the B.C.'s are

$$\left.\begin{aligned} &u_g \text{ or } N_x \\ &v_g \text{ or } N_y \\ &\beta \text{ or } M \end{aligned}\right\} \text{prescribed at each end}$$

for the Marguerre formulation

$$\left.\begin{aligned} &u_g \text{ or F} \\ &v_g \text{ or } V + y,_x F \\ &\beta \text{ or } M \end{aligned}\right\} \text{prescribed at each end}$$

Example



$$y = \frac{1}{2}ax^2$$

$$y_{,x} = ax$$

$$y_{,xx} = a$$

$$a = \frac{2H}{L^2}$$

$$y_{,x} = aL\frac{x}{L} = \left(2\frac{H}{L}\right)\frac{x}{L}$$

x goes from 0 to 1; then $2H/L$ provides a measure of $y_{,x}$.

The maximum value of $y_{,x}$ is $2H/L$.

$$\alpha = \left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}^{1/2}$$

using $\theta = x$

$$\alpha = \{1 + (y_{,x})^2\}^{1/2}$$

$$\alpha_{max} = \left\{1 + \left(\frac{2H}{L}\right)^2\right\}^{1/2}$$

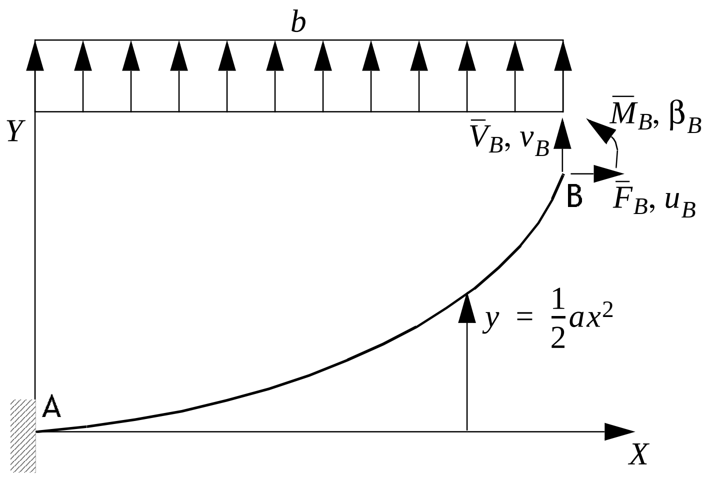
Suppose $H = L/10$ then $\alpha_{max} = 1.02$. Thus, it is reasonable to assume α is constant and equal to 1. In general, when $(y_{,x})^2$ is small wrt 1, one can simplify these equations. This is what we call the "shallow cartesian formulation." The resulting approximate equations are:

$$\alpha \cong 1 \qquad \frac{1}{R} \cong y_{,xx}$$

$$\vec{t}_1 \cong \vec{i} + y_{,x}\vec{j} \qquad \vec{t}_2 \cong -y_{,x}\vec{i} + \vec{j}$$

$$\cos\psi \cong 1 \qquad \sin\psi \cong y_{,x}$$

2.4.5 Shallow Cartesian Solution - Parabolic Geometry



Use Marguerre formulation, assume thin and shallow

$$b_x = 0\ ,\ b_y = \text{constant} = b\ ,\ m_g = 0$$

$$y,_x = ax$$

at $x = 0$ $\quad u_g = v_g = \beta = 0$ (fixed support)

at $x = L$ $\quad N_x = \bar{N}_{Bx}$

$$N_y = \bar{N}_{By}$$

$$M = \bar{M}_B$$

Governing equations

Equilibrium

$$F,_x = 0$$

$$V,_x + (Fax),_x + b = 0$$

$$M,_x + V = 0$$

Force Deformation (set $u_g = u$; $v_g = v$)

$$\frac{F}{D_S} = u,_x + axv,_x$$

$$\frac{V}{D_T} = v,_x - \beta$$

$$\frac{M}{D_B} = \beta,_x$$

Boundary Conditions

at $x = L$ $\qquad$ at $x = 0$

$$F = \bar{N}_{Bx} \qquad u = v = \beta = 0$$

$$V = \bar{N}_{By} - aLF$$

$$M = \bar{M}_B$$

Solution for the internal forces

$$F,_x = 0$$

$$F = C_1 = \bar{N}_{Bx}$$

$$V,_x + (\bar{N}_{Bx}ax),_x + b = 0$$

$$V = -bx - (ax)\bar{N}_{Bx} + C_2$$

$$\bar{N}_{By} - aLF = -bx - (ax)\bar{N}_{Bx} + C_2$$

$$C_2 = \bar{N}_{By} + bL$$

$$V = -bx - (ax)\bar{N}_{Bx} + \bar{N}_{By} + bL$$

$$V = \bar{N}_{By} + b(L-x) - (ax)\bar{N}_{Bx}$$

$$M = \bar{N}_{By}x + b\left(Lx - \frac{x^2}{2}\right) - \left(a\frac{x^2}{2}\right)\bar{N}_{Bx} + C_3$$

$$\bar{M}_B = \bar{N}_{By}L + b\frac{L^2}{2} - a\frac{L^2}{2}\bar{N}_{Bx} + C_3$$

$$C_3 = \bar{M}_B - \bar{N}_{By}L - b\frac{L^2}{2} + a\frac{L^2}{2}\bar{N}_{Bx}$$

$$M = \frac{b}{2}(L-x)^2 - \frac{a\bar{N}_{Bx}}{2}(L^2 - x^2) + (L-x)\bar{N}_{By} + \bar{M}_B = D_B\beta,_x$$

Solution for displacements

$$D_B\beta = -\frac{b}{6}(L-x)^3 + \frac{a\bar{N}_{Bx}}{2}\left(-xL^2 + \frac{x^3}{3}\right) + \left(Lx - \frac{x^2}{2}\right)\bar{N}_{By} + x\bar{M}_B + C_4$$

$$C_4 = \frac{b}{6}L^3$$

$$v,_x = \frac{V}{D_T} + \beta$$

$$v = \frac{1}{D_T}\left\{x\bar{N}_{By} + b\left(Lx - \frac{x^2}{2}\right) - \left(a\frac{x^2}{2}\right)\bar{N}_{Bx}\right\}$$

$$+ \frac{1}{D_B}\left\{b\left(L^2\frac{x^2}{4} - L\frac{x^3}{6} + \frac{x^4}{24}\right) + \frac{a\bar{N}_{Bx}}{2}\left(-\frac{x^2}{2}L^2 + \frac{x^4}{12}\right) + \left(L\frac{x^2}{2} - \frac{x^3}{6}\right)\bar{N}_{By} + \frac{x^2}{2}\bar{M}_B\right\} + C_5$$

$$C_5 = 0$$

$$u_{,x} = \frac{F}{D_S} - axv_{,x}$$

$$u_{,x} = \frac{F}{D_S} - (ax)\left(\frac{V}{D_T} + \beta\right)$$

$$u = \frac{\bar{N}_{Bx}}{D_S}x + \frac{1}{D_T}\left\{-\frac{ax^2}{2}\bar{N}_{By} + \frac{a^2x^3}{3}\bar{N}_{Bx} + b\left(-\frac{aLx^2}{2} + \frac{ax^3}{3}\right)\right\}$$

$$+ \frac{1}{D_B}\left\{-ab\left(L^2\frac{x^3}{6} - L\frac{x^4}{8} + \frac{x^5}{30}\right) + \frac{a^2\bar{N}_{Bx}}{2}\left(\frac{x^3}{3}L^2 - \frac{x^5}{15}\right) + a\bar{N}_{By}\left(-L\frac{x^3}{3} + \frac{x^5}{8}\right) + \frac{ax^3}{3}\bar{M}_B\right\} + C_6$$

$$C_6 = 0$$

Flexibility matrix - cartesian formulation

Set

$$\underline{u}_B = \begin{bmatrix} u_{gB} \\ v_{gB} \\ \beta_B \end{bmatrix} \qquad \underline{F}_B = \begin{bmatrix} N_{Bx} \\ N_{By} \\ M_B \end{bmatrix}$$

$$\underline{u}_B = \underline{u}_{B,o} + \underline{T}_{AB}\underline{u}_A + \underline{f}_B\underline{F}_B$$

$$\underline{T}_{AB} = \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & L \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{N}_{Ax} = \bar{N}_{Ax,o} - \bar{N}_{Bx}$$

$$\bar{N}_{Ay} = \bar{N}_{Ay,o} - \bar{N}_{By}$$

$$\bar{M}_A = \bar{M}_{A,o} - \bar{M}_B - L\bar{N}_{By} + h\bar{N}_{Bx}$$