Problem Set 3

Due: Thursday, October 19

Problem 1

(a) If Bo did not have to wait, then he arrived while the system was empty. For an M/M/1 queueing system, the probability to be in the state with no customer is:

$P_0 = 1 - \rho$ with $\rho = \dfrac{\lambda}{\mu}$.

If Alvin had to wait, then Bo was still in service when he arrived, since no customer can go into the building between Bo and Alvin's arrivals. The probability that Bo's service is longer than 2 minutes is $e^{-2\mu}$. Therefore, the probability that Bo did not have to wait at all for the service but that Alvin did have to wait is:

$$P = \left(1 - \frac{\lambda}{\mu}\right) e^{-2\mu} = 0.2696$$

(b) The probability that Bo finds 8 customers is the steady state probability of their being 8 customers in an M/M/1 system at a random point in time. This is just $\rho^8(1-\rho)$. If only one customer is in service when Alvin arrives, then it must be Bo. This means that in the two minutes between Bo and Alvin's arrivals exactly 8 departures must have occurred from the system. Since we are talking about the departure process of customers already in the system we can think of them as departing according to a Poisson process of rate μ.

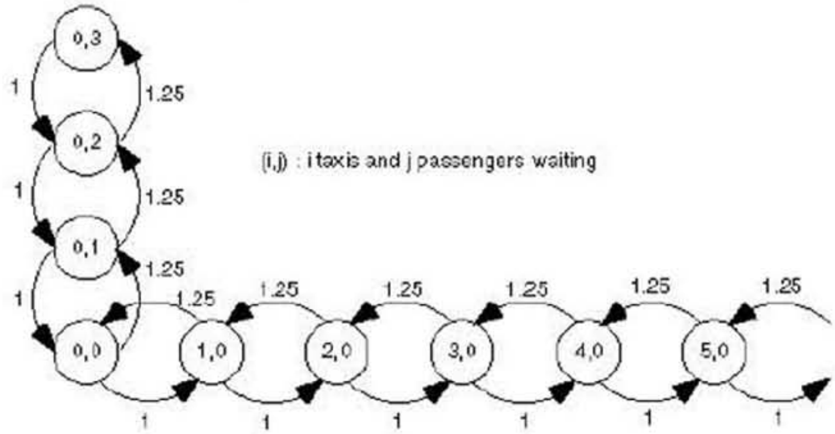
Therefore the probability of exactly eight customers leaving during the two minutes is the probability of having 8 poisson occurances in two minutes which is given by: $\dfrac{(2\mu)^8 e^{-2\mu}}{8!}$. Plugging in our known values of $\lambda = 0.4$ customers per minute and $\mu = 1$ customer per minute. We get that the probability of the event described in part (b) is:

$$\rho^8(1-\rho)\frac{(2\mu)^8 e^{-2\mu}}{8!} = 3.39 \cdot 10^{-7}$$

Problem 2

By Armann Ingolfsson '93

The state transition diagram is



Note that if there are taxis waiting, then no passengers will have to wait, and if there are passengers waiting, it must be because there are no taxis available. Thus, states that have both $i > 0$ and $j > 0$ are not possible.

The balance equations are

$$\begin{aligned}
\pi_{0,3}(1) &= \pi_{0,2}(1.25) \Rightarrow \pi_{0,2} = (0.8)\pi_{0,3} \\
\pi_{0,2}(1) &= \pi_{0,1}(1.25) \Rightarrow \pi_{0,1} = (0.8)\pi_{0,2} = (0.8)^2\pi_{0,3} \\
\pi_{0,1}(1) &= \pi_{0,0}(1.25) \Rightarrow \pi_{0,0} = (0.8)^3\pi_{0,3} \\
&\vdots \\
\pi_{i-1,0}(1) &= \pi_{i,0}(1.25) \Rightarrow \pi_{i,0} = (0.8)\pi_{i-1,0} = (0.8)^{i+3}\pi_{0,3}
\end{aligned}$$

Since $\sum_{i,j} \pi_{i,j} = 1$, we have:

$$\pi_{0,3} + \pi_{0,2} + \pi_{0,1} + \sum_{i=1}^{\infty} \pi_{i,0} = \pi_{0,3} \sum_{i=0}^{\infty} (0.8)^i = \pi_{0,3} \frac{1}{1-0.8} = 1 \Rightarrow \pi_{0,3} = 0.2$$

$$\pi_{0,3} = 0.2,\ \pi_{0,2} = (0.8)(0.2),\ \pi_{0,1} = (0.8)^2(0.2),\ \pi_{i,0} = (0.8)^{i+3}(0.2) \text{ for } i = 0, 1, 2, \ldots$$

(a)

$$\begin{aligned}
\text{E}[\#\text{ of taxis waiting}] &= \sum_{i=1}^{\infty} i\pi_{i,0} = \sum_{i=1}^{\infty} i(0.8)^{i+3}(0.2) = (0.8)^4 \sum_{i=1}^{\infty} i(0.8)^{i-1}(0.2) \\
&= (0.8)^4 \text{E}[\text{geometric r.v. with } p = 0.2] \\
&= (0.8)^4(1/0.2) = \left(\frac{4}{5}\right)^4 5 = \frac{256}{125} \simeq 2.05
\end{aligned}$$

(b)

$$
\begin{aligned}
\text{E}[\#\text{ of passengers waiting}] &= \sum_{j=1}^{3} j\pi_{0,j} = 1\pi_{0,1} + 2\pi_{0,2} + 3\pi_{0,3} \\
&= (0.8)^2(0.2) + 2(0.8)(0.2) + 3(0.2) = \frac{16}{125} + \frac{8}{25} + \frac{3}{5} \\
&= \frac{131}{125} \simeq 1.05
\end{aligned}
$$

(c) Let N be the number of passengers that leave in one hour because they arrive when there is no more room, i.e., the system is in state $(0,3)$, and let T be the amount of time in one hour that the system is in state $(0,3)$. Assuming steady state, we have

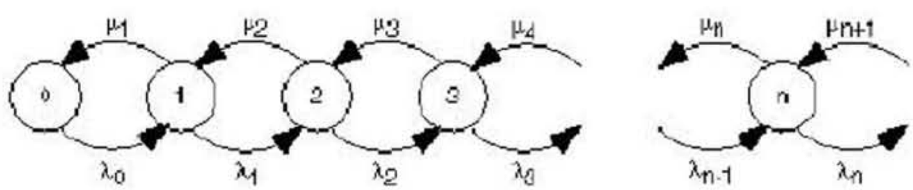
$$
\begin{aligned}
\text{E}[N] &= \int_0^{60} \text{E}[N|T=t] f_T(t)dt = \int_0^{60} \lambda t f_T(t)dt = \lambda \int_0^{60} t f_T(t)dt \\
&= \lambda \text{E}[T] = (1.25)(\pi_{0,3})(60) = (1.25)(0.2)(60) = 15 \text{ passengers}
\end{aligned}
$$

Problem 3

By Armann Ingolfsson '93

To be consistent with the notation of the problem, we'll use $\{P_i\}$ for the steady state probabilities, instead of $\{\pi_i\}$ as in the previous problem.

The state transition diagram for any Markovian (i.e., memoryless) queue looks as follows



The only difference between such a queue and an M/M/1 queue is that here the arrival and service rates are allowed to depend on the state of the system. The balance equations are

$$
P_i\lambda_i = P_{i+1}\mu_{i+1} \Rightarrow P_{i+1} = \frac{\lambda_i}{\mu_{i+1}} P_i \text{ for } i = 0, 1, 2, \ldots
$$

$$
\Rightarrow P_i = \frac{\lambda_{i-1}\lambda_{i-2}\cdots\lambda_0}{\mu_i\mu_{i-1}\cdots\mu_1} P_0 \text{ for } i = 1, 2, \ldots
$$

The equation $\sum_0^\infty P_i = 1$ can then be used to solve for P_0.

(a) Applying the general formula, we have

$$P_i = \frac{(\lambda/i)(\lambda/(i-1))\cdots(\lambda)}{(\mu)(\mu)\cdots(\mu)} P_0 = \frac{(\lambda/\mu)^i}{i!} P_0 \text{ for } i = 0, 1, \ldots \tag{1}$$

Then

$$\sum_{i=0}^{\infty} P_i = P_0 \sum_{i=0}^{\infty} \frac{(\lambda/\mu)^i}{i!} = P_0 e^{\lambda/\mu} = 1 \Rightarrow P_0 = e^{-\lambda/\mu}$$

$$\Rightarrow P_i = \frac{(\lambda/\mu)^i}{i!} e^{-\lambda/\mu} \text{ for } i = 0, 1, \ldots$$

So we see that in steady state, the number of customers in the system follows a Poisson pmf, with mean λ/μ. The fraction of time the server is busy, ρ, is equal to $1 - P_0 = 1 - e^{-\lambda/\mu}$. The system reaches steady-state as long as

$$\rho < 1 \Leftrightarrow 1 - e^{-\lambda/\mu} < 1 \Leftrightarrow e^{-\lambda/\mu} > 0 \Leftrightarrow \lambda/\mu < \infty$$

So all we need to require is that $\lambda < \infty$ and $\mu > 0$.

(c) Since the pmf for the steady state number of people in the system is Poisson with mean λ/μ, we have $\bar{L} = \lambda/\mu$. By Little's law, we have $\bar{L} = \bar{\lambda}\bar{W}$, being careful to remember that we must use the *average* arrival rate $\bar{\lambda}$. The average arrival rate can be computed as

$$\begin{aligned}
\bar{\lambda} &= \sum_{i=0}^{\infty} \lambda_i P_i = \sum_{i=0}^{\infty} \frac{\lambda}{i+1} \frac{(\lambda/\mu)^i}{i!} e^{-\lambda/\mu} \\
&= \mu e^{-\lambda/\mu} \sum_{i=0}^{\infty} \frac{(\lambda/\mu)^{i+1}}{(i+1)!} = \mu e^{-\lambda/\mu} \sum_{i=1}^{\infty} \frac{(\lambda/\mu)^i}{i!} \\
&= \mu e^{-\lambda/\mu}(e^{\lambda/\mu} - 1) = \mu(1 - e^{-\lambda/\mu})
\end{aligned}$$

Hence,

$$\bar{W} = \bar{L}/\bar{\lambda} = \frac{\lambda/\mu}{\mu(1 - e^{-\lambda/\mu})} = \frac{\lambda}{\mu^2(1 - e^{-\lambda/\mu})}$$

A tempting, but *wrong* way to compute $\bar{W}$ is to say

$$\bar{W} \overset{?}{=} \sum_{k=0}^{\infty} \left(\frac{k+1}{\mu}\right) P_k = \frac{1}{\mu}(\bar{L} + 1) = \frac{1}{\mu}(\frac{\lambda}{\mu} + 1)$$

The reason that this approach is wrong is that the probability that a randomly chosen customer arrives when there are k customers present in the system, say Q_k, is *not* equal to the steady state probability P_k that there are k customers in the system. In fact,

$$\begin{aligned} Q_k &= \Pr\{k \text{ customers present when randomly chosen customer arrives}\} \\ &= \frac{\#\text{ who arrive while there are } k \text{ present}}{\text{total } \#\text{ of customers who arrive over a long period } [0,T]} \\ &= \frac{(\text{length of time there are } k \text{ present}) \times \lambda_k}{\bar{\lambda} T} \\ &= \frac{T P_k \lambda_k}{\bar{\lambda} T} = \frac{\lambda_k}{\bar{\lambda}} P_k \end{aligned}$$

Using Q_k instead of P_k in the summation above, we get the correct answer. There is a theorem, called PASTA (Poisson Arrivals See Time Averages), which states that if the customer arrival process is Poisson, then $Q_k = P_k$. In this case, the arrival process is not Poisson, because the arrival rate changes with the state of the system.

Problem 4

By Armann Ingolfsson '93

$$N(t) \equiv \#\text{ of broken down buses at time } t$$

Then

$$\Pr\{N(t+\Delta t) = N(t) + 1\} = 1\Delta t + o(\Delta t)$$

(buses break down at a rate of one per hour) and

$$\Pr\{N(t+\Delta t) = N(t) - 1\} = \mu(N(t))\Delta t + o(\Delta t)$$

where the function $\mu(N(t))$ is determined by:

(1) The total number of mechanics employed, k, and

(2) The assignment of mechanics to the $N(t)$ broken down buses.

The problem is to make decisions (1) and (2) to minimize expected cost per hour, E$[C]$. The cost C per hour consists of

$$\begin{aligned} C &= \text{wages } + \text{ cost of buses not in service} \\ &= \$10k + \$40 \int_0^1 N(t)dt \end{aligned}$$

In steady state, we have

$$\mathrm{E}\left[\int_0^1 N(t)dt\right] = 1 \times \bar{N}$$

where $\bar{N}$ is the average number of broken down buses, so

$$\mathrm{E}[C] = 10k + 40\bar{N}$$

The number of buses $N(t)$ behaves according to a discrete state Markov process with a transition diagram which is identical to the one in the last problem, with $\lambda_i = 1$ and μ_i to be determined.

(i): Service rate proportional to k. Suppose that k, the total number of mechanics, has been decided upon. Recall that the steady state probabilities for $N(t)$ will be

$$P_i = \frac{1}{\mu_1 \mu_2 \cdots \mu_i} P_0$$

Suppose I were to increase service rate μ_j. Then $P_0, P_1, \ldots, P_{j-1}$ would increase, and $P_j, P_{j+1}, \ldots$ would decrease, so $\bar{N} = \sum i P_i$ would decrease. Since I would like to minimize $\bar{N}$, it must be optimal to make all service rates μ_i as large as possible. What this means is that whenever one or more buses are broken, all mechanics will be working, so $\mu_i = (1/2)k$ for $i = 1, 2, \ldots$. Note that it makes no difference whether all k mechanics finish reparing the first bus to break down before starting work on the second bus to break down, or whether they split up when the second bus breaks down. Since μ_i is constant, this is equivalent to an M/M/1 queue, for which $\bar{N} = \lambda/(\mu - \lambda) = 1/(k/2 - 1) = 2/(k-2)$ (thus, k has to be at least three for stability), so

$$\mathrm{E}[C] = 10k + \frac{80}{k-2}$$

Differentiating with respect to k and setting equal to zero we get

$$10 - \frac{80}{(k-2)^2} = 0 \Rightarrow k = 2 + \sqrt{8} \simeq 4.8$$

Thus the optimal number of mechanics must be either four or five. We have

$$\mathrm{E}[C]|_{k=4} = 80 \text{ and } \mathrm{E}[C]|_{k=5} = 76\frac{2}{3}$$

so $k^* = 5$ minimizes cost per hour.

(ii): Service rate proportional to $\sqrt{k}$. In this case, it does make a difference whether all mechanics work on the same bus, or split up to work on several buses. For example, if $k = 4$ mechanics are available and 2 buses are in for repair, then the overall service rate depends on how the mechanics are assigned to buses as follows:

Assignment	Overall service rate, μ_2
$4+0$	$\sqrt{4}(1/2) = 1$
$3+1$	$(\sqrt{3}+\sqrt{1})(1/2) = 1.366$
$2+2$	$(\sqrt{2}+\sqrt{2})(1/2) = 1.414$

So for this case, it is best to assign two mechanics to each bus. One could perform this analysis for $k = 1, 2, \ldots$ and $i = \#$ of buses $= 1, 2, \ldots$, find the best assignment for each case, and compute $\mathrm{E}[C]$ for each k. The answer turns out to be that $k^* = 5$ again minimizes expected cost per hour at \$ 94.20. Rather than go through this tedious analysis, suppose we assume that the crew cannot be divided up. Then the service rate will be $\mu_i = (1/2)\sqrt{k}$, and

$$\bar{N} = \lambda/(\mu - \lambda) = 1/(\sqrt{k}/2 - 1) = 2/(\sqrt{k} - 2)$$

Now we see that $k > 4$ is required for stability, and

$$\mathrm{E}[C] = 10k + \frac{80}{\sqrt{k}-2} \Rightarrow \frac{\partial \mathrm{E}[C]}{\partial k} = 10 - \frac{80}{(\sqrt{k}-2)^2}\frac{1}{2\sqrt{k}} = 0 \Rightarrow \sqrt{k}(\sqrt{k}-2)^2 = 4$$

By trial and error, we find that $k \simeq 9.8$. Since

$$\mathrm{E}[C]|_{k=9} = 170 \text{ and } \mathrm{E}[C]|_{k=10} = 168.83$$

the optimal crew size is $k^* = 10$.

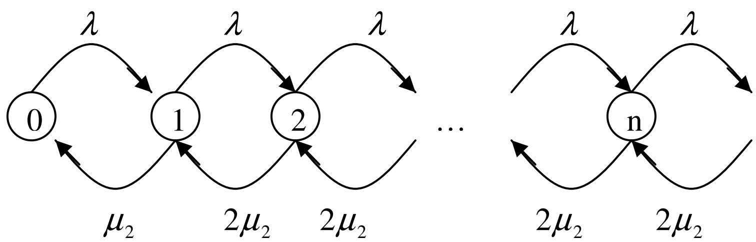
Problem 5

For each scenario, the total cost per minute is $TC_i = 1 \cdot L_i + C_i$ with $i = 1 \; or \; 2$. We have to find the expected number of cars for each scenario.

If there is only one server, the system can be modeled as an M/M/1 system with infinite capacity. According to the formula derived in class we have

$$L_1 = \frac{\lambda}{\mu_1 - \lambda} = \frac{1/40}{1/15 - 1/40} = 0.6.$$

If there are two parallel servers, the system can be modeled as an M/M/2 system with infinite capacity. The corresponding state transition diagram is the following:



$$\begin{cases} \lambda P_0 = \mu_2 P_1 \\ \lambda P_1 = 2\mu_2 P_2 \\ \lambda P_2 = 2\mu_2 P_3 \\ \vdots \\ \lambda P_{n-1} = 2\mu_2 P_n \end{cases} \quad \text{thus} \quad \begin{cases} P_1 = \dfrac{\lambda}{\mu_2} P_0 \\ P_n = \left(\dfrac{\lambda}{2\mu_2}\right)^{n-1} P_1 \quad for\ n \geq 1 \end{cases}$$

$$\sum_{i=0}^{\infty} P_i = 1 \quad \Rightarrow \quad P_0 = \frac{1 - \dfrac{\lambda}{2\mu_2}}{1 + \dfrac{\lambda}{2\mu_2}} = \frac{5}{11}$$

Now, we can compute the expected number of cars in the system:

$$L_2 = \sum_{n=1}^{\infty} nP_n = P_1 \sum_{n=1}^{\infty} n\left(\frac{\lambda}{2\mu_2}\right)^{n-1}$$

Let's have $\alpha = \dfrac{\lambda}{2\mu_2}$, then

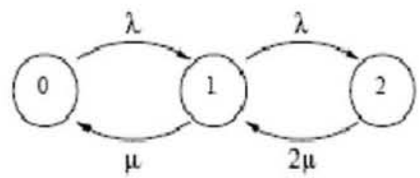
$$L_2 = P_1 \sum n\alpha^{n-1} = P_1 \frac{d}{d\alpha}\left(\sum_{n=1}^{\infty} \alpha^n\right) = P_1 \frac{d}{d\alpha}\left(\frac{\alpha}{1-\alpha}\right) = P_1 \frac{1}{(1-\alpha)^2}$$

Therefore $L_2 = \dfrac{1}{\left(1 - \dfrac{\lambda}{2\mu_2}\right)^2} \dfrac{\lambda}{\mu} P_0 = 0.87$

If the two total costs are equal, then : $C_1 - C_2 = L_2 - L_1 = 0.27 \quad \Rightarrow \quad C_1 \succ C_2$

Problem 6

(a) The state transition diagram of this M/M/2 queueing system is



The balance equations and the normalization equation are

$$\lambda P_0 = \mu P_1$$
$$\lambda P_1 = 2\mu P_2$$
$$P_0 + P_1 + P_2 = 1$$

$P_1 = \frac{\lambda}{\mu} P_0 = \rho P_0$. $P_2 = \frac{\lambda}{2\mu} P_1 = \frac{1}{2}\rho P_1 = \frac{1}{2}\rho^2 P_0$. Using the normalization equation,

$$P_0 + \rho P_0 + \frac{1}{2}\rho^2 P_0 = P_0\left(1 + \rho + \frac{1}{2}\rho^2\right) = 1 \quad \Rightarrow \quad P_0 = \frac{1}{1 + \rho + \frac{1}{2}\rho^2}$$

The expected number of men who are busy serving a customer at any given time is given by

$$1 \times P_1 + 2 \times P_2 = \frac{\rho}{1 + \rho + \frac{1}{2}\rho^2} + \frac{\rho^2}{1 + \rho + \frac{1}{2}\rho^2} = \frac{\rho + \rho^2}{1 + \rho + \frac{1}{2}\rho^2}$$

(b) Using the data collected, we have the following equation:

$$\frac{\rho + \rho^2}{1 + \rho + \frac{1}{2}\rho^2} = \frac{8,000}{10,000} = 0.8 \quad \Rightarrow \quad 0.8 + 0.8\rho + 0.4\rho^2 = \rho + \rho^2$$
$$\Rightarrow \quad 0.6\rho^2 + 0.2\rho - 0.8 = 0$$
$$\Rightarrow \quad \rho^2 + \frac{1}{3}\rho - \frac{4}{3} = 0$$

It gives $\rho = 1$ (the other root, $-\frac{4}{3}$, is meaningless). Note that the actual arrival rate of customers is $\lambda' = \lambda(1 - P_2)$. Since 40,000 customers received service during 10,000 hours,

$$\lambda(1 - P_2) = \frac{40,000}{10,000} = 4$$

Since $\rho = 1$, we have $P_2 = \frac{\frac{1}{2}\rho^2}{1+\rho+\frac{1}{2}\rho^2} = \frac{1}{5}$. Therefore

$$\lambda = \frac{4}{(1 - P_2)} = \frac{4}{4/5} = 5$$

The number of customers lost during these 10,000 hours is

$$\lambda P_2 \times 10,000 = 5 \times \frac{1}{5} \times 10,000 = 10,000$$

Problem 7

(a) The state, k, can be thought of as the number of service phases. It is not equal to the number of customers in the system. The system is busy when in states 1 or 2. Hence,

$$\text{server utilization factor} = \frac{P_1 + P_2}{P_0 + P_1 + P_2}$$

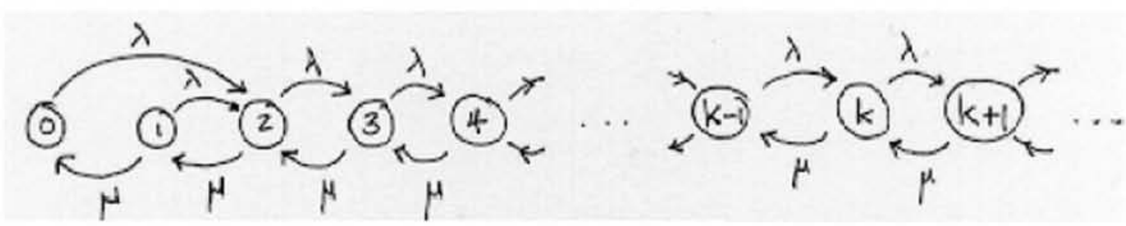
The steady state equations are:

$$\begin{aligned} P_0 \cdot \lambda &= P_1 \cdot \mu \\ P_1 \cdot \mu &= P_2 \cdot \mu \\ P_0 \cdot \lambda &= P_2 \cdot \mu \\ P_0 + P_1 + P_2 &= 1 \end{aligned}$$

This implies that the service utiilization factor

$$\rho = \frac{\frac{\lambda}{\mu} \cdot P_0 + \frac{\lambda}{\mu} \cdot P_0}{P_0 + \frac{\lambda}{\mu} \cdot P_0 + \frac{\lambda}{\mu} \cdot P_0} = \frac{2\lambda}{\mu + 2\lambda}$$

(b) Again, k is the number of service phases to be completed.



(c) The steady-state equations are:

$$\begin{aligned} P_1 &= \frac{\lambda}{\mu} P_0 \\ P_2 &= \left(1 + \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right) P_0 \\ P_3 &= \left(1 + \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^2 P_0 \\ &\vdots \\ P_k &= \left(1 + \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{k-1} P_0 \end{aligned}$$

and $\sum_{k=0}^{\infty} \mathrm{P_k} = 1$. Solving for $\mathrm{P_0}$ in this last equation, assuming $\frac{\lambda}{\mu} < 1$, we get

$$\mathrm{P_0} = \frac{1 - \frac{\lambda}{\mu}}{1 + \frac{\lambda}{\mu}}$$

Server utilization is then

$$1 - \mathrm{P_0} = \frac{2\lambda}{\mu + \lambda}$$

(d) The fraction of customers who experience the Erlang order 2 service time is the fraction of customers who arrive to an empty system. Hence this fraction is equal to $\mathrm{P_0}$.

(e) If a customer finds the system in state 0, then the service time has a mean of $\frac{2}{\mu}$ minutes. For all the other states, the service time has a negative exponential probability density function with mean $\frac{1}{\mu}$. Therefore, we have two cases to consider:

$$E[T_{service}] = P_0 \frac{2}{\mu} + (1 - P_0)\frac{1}{\mu} = \frac{2}{\mu + \lambda}$$

<u>Extra Question:</u>

Given k service phases to be completed, W_q is equal to $\frac{k}{\mu}$. Hence,

$$\begin{aligned}
W_q &= \sum_{k=0}^{\infty} (W_q|k)\mathrm{P_k} = \sum_{\mathrm{k}=0}^{\infty} \frac{\mathrm{k}}{\mu}\mathrm{P_k} = \frac{1}{\mu}\sum_{\mathrm{k}=0}^{\infty} \mathrm{kP_k} \\
&= \frac{1}{\mu}\mathrm{P_0}\left[1 \cdot \left(\frac{\lambda}{\mu}\right) + 2 \cdot \left(1 + \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right) + 3 \cdot \left(1 + \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^2 + \cdots\right] \\
&= \frac{1}{\mu}\mathrm{P_0}\left[\left(1 + \frac{\lambda}{\mu}\right)\left(1 + 2(\frac{\lambda}{\mu}) + 3(\frac{\lambda}{\mu})^2 + \cdots\right) - 1\right] \\
&= \frac{1}{\mu}\left(\frac{1 - \frac{\lambda}{\mu}}{1 + \frac{\lambda}{\mu}}\right)\left[\left(1 + \frac{\lambda}{\mu}\right)\left(\frac{\frac{\lambda}{\mu}}{(1 - \frac{\lambda}{\mu})^2}\right) - 1\right] \\
&= \frac{1}{\mu}\left[\left(\frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}}\right) - 1\right] = \frac{1}{\mu}\left(\frac{\lambda}{\mu - \lambda} - 1\right)
\end{aligned}$$

The other quantities follow suite:

$$\begin{aligned} L_q &= \lambda W_q \\ W &= W_q + W_{SF} = W_q + \frac{2}{\mu + \lambda} \\ L &= \lambda W \end{aligned}$$