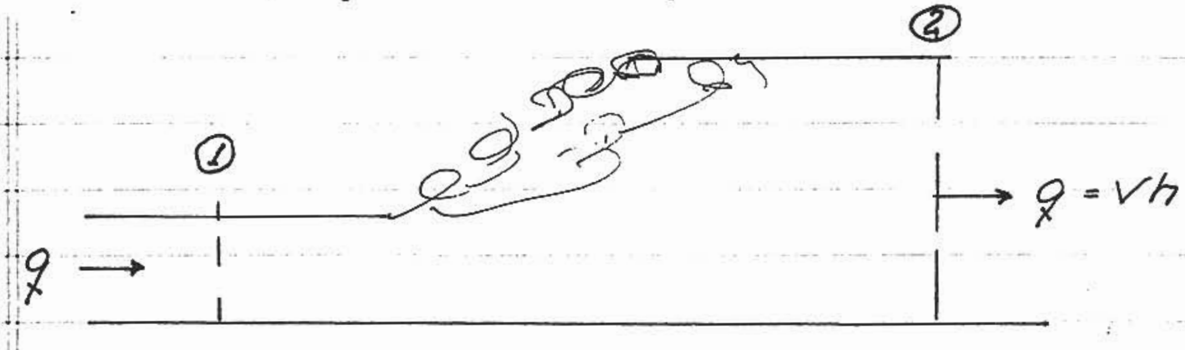
LECTURE #28

1.060 ENGINEERING MECHANICS II

The Unassisted Hydraulic Jump in a Rectangular Channel

With channel width $= b =$ constant, we treat problem in terms of "per unit width." $q = Q/b =$ discharge $= Vh =$ const.



$$MP_1 = \tfrac{1}{2}\rho g h_1^2 + \rho V_1^2 h_1 = MP_2 = \tfrac{1}{2}\rho g h_2^2 + \rho V_2^2 h_2$$

Divide by

$$1 + 2\underbrace{V_1^2 h_1^2}_{q^2}/(g h_1^2)h_1 = (h_2/h_1)^2 + 2\underbrace{V_2^2 h_2^2}_{q^2}/(g h_1^2)h_2$$

$$2\left(\frac{q^2}{g h_1^3} - \frac{q^2}{g h_1^2 h_2}\right) = 2\frac{q^2}{g h_1^3}\left(1 - \frac{h_1}{h_2}\right) = \left(\frac{h_2}{h_1}\right)^2 - 1 =$$

$$\left(\frac{h_2}{h_1} - 1\right)\left(\frac{h_2}{h_1} + 1\right) = \frac{h_2}{h_1}\left(1 - \frac{h_1}{h_2}\right)\left(\frac{h_2}{h_1} + 1\right)$$

or

$$\frac{q^2}{g h_1^3} = \frac{V_1^2}{g h_1} = Fr_1^2 = \frac{1}{2}\,\frac{h_2}{h_1}\left(\frac{h_2}{h_1} + 1\right)$$

Quadratic Eq. in h_2/h_1:

$$h_2/h_1 = \tfrac{1}{2}\left(-1 \pm \sqrt{1 + 8Fr_1^2}\right) \quad \text{(JUMP CONDITION)}$$

Multiply Fr_1^2-eq. by $(h_1/h_2)^3$: $Fr_2^2 = \frac{1}{2}\frac{h_1}{h_2}\left(1 + \frac{h_1}{h_2}\right)$

$$h_1/h_2 = \tfrac{1}{2}\left(-1 + \sqrt{1 + 8Fr_2^2}\right)$$

h_1 = upstream depth
h_2 = downstream depth $\Big\}$ h_1 & h_2 are <u>conjugate depths</u>

<u>Head Loss across Hydraulic Jump</u>

$$\Delta H_j = H_1 - H_2 = E_1 - E_2 = h_1 - h_2 + \frac{q^2}{2gh_1^2} - \frac{q^2}{2gh_2^2}$$

$$\Delta H_j = h_1 - h_2 + \frac{q^2}{2gh_1^2}\left(1 - \left(\frac{h_1}{h_2}\right)^2\right) = h_1 - h_2 + \frac{h_1}{2} Fr_1^2 \left(1 - \left(\frac{h_1}{h_2}\right)^2\right)$$

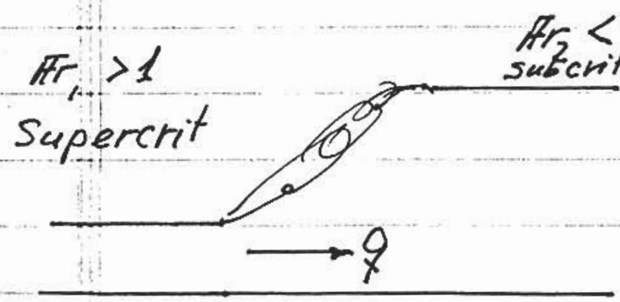
but $Fr_1^2 = \frac{1}{2}\frac{h_2}{h_1}\left(1 + \frac{h_2}{h_1}\right)$ from Momentum, so

$$\underline{\Delta H_j} = h_1 - h_2 + \frac{1}{4} h_1 \frac{h_2}{h_1}\left(1 + \frac{h_2}{h_1}\right)\left(1 - \left(\frac{h_1}{h_2}\right)^2\right) =$$

$$-(h_2 - h_1) + \frac{1}{4}\frac{1}{h_1 h_2}(h_1 + h_2)(h_2 + h_1)(h_2 - h_1) =$$

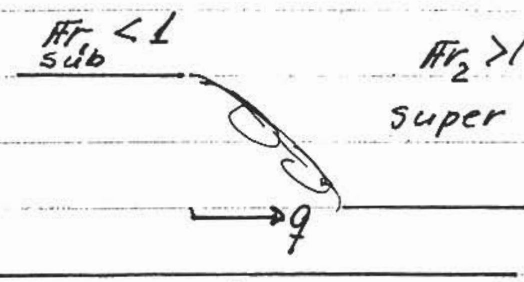
$$(h_2 - h_1)\left\{\frac{1}{4h_1h_2}(h_1 + h_2)^2 - 1\right\} = \underline{\frac{(h_2 - h_1)^3}{4h_1h_2}}$$

Derivation was completely general, but we must require that $H_2 \leq H_1$, i.e. $\Delta H_j \geq 0$
<u>Only possible if $h_2 > h_1$</u>



$\Delta H_j > 0$

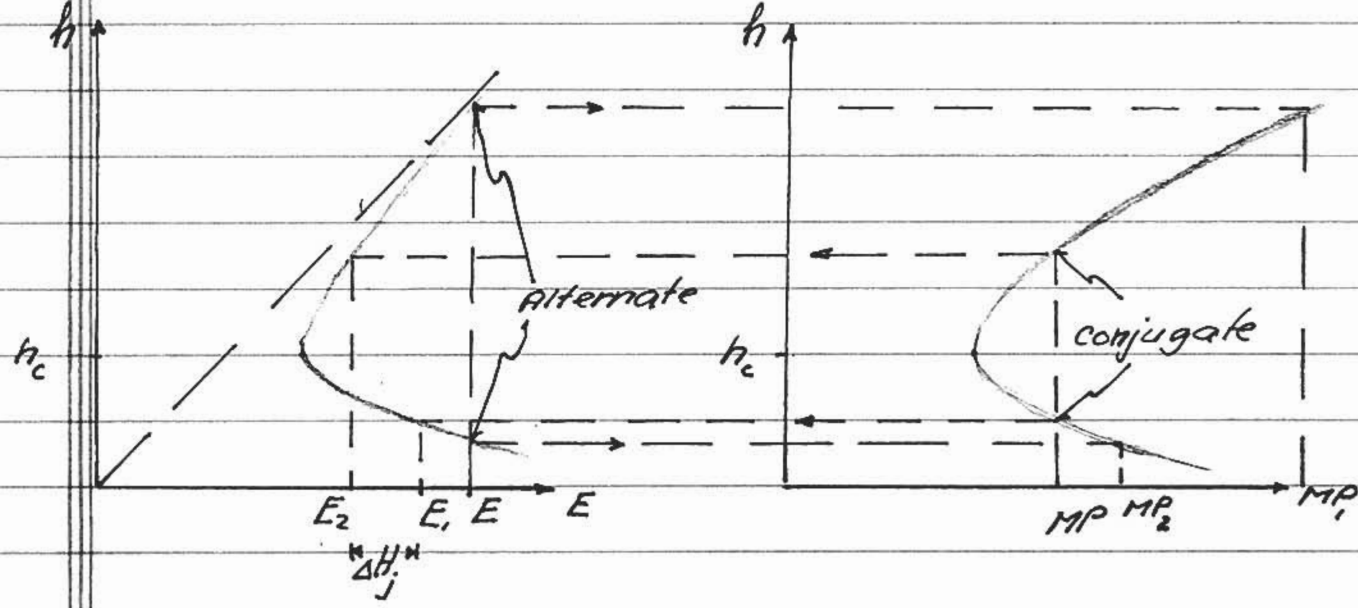
TRANSITION IS POSSIBLE FROM SUPER TO SUBCRIT.



$\Delta H_j < 0$

PHYSICALLY IMPOSSIBLE TO HAVE SUB TO SUPER TRANSITION

Combined use of E-h & MP-h Diagrams



Jump from supercritical to subcritical is possible since the flow is expanding and is associated with a loss in head. $MP_1 = MP_2 \Rightarrow E_1 > E_2 \Rightarrow$ There is a headloss

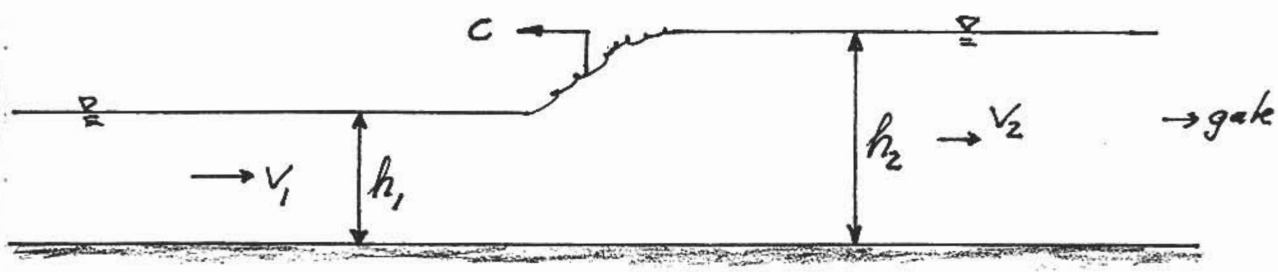
It is possible to have a flow going from subcritical to supercritical without violating energy conservation, e.g. flow under a gate, but only if there is an exterior force applied to the fluid, e.g. pressure force from the gate. $E_1 = E_2 \Rightarrow MP_1 > MP_2 \Rightarrow MP_2$ needs help to balance MP_1

Supercritical Flow: M is important in MP, $V^2/2g$ is important in E

Subcritical Flow: P is important in MP, h is important in E.

The Moving Hydraulic Jump (The Bore)

Imagine a steady uniform flow in a rectangular channel, into which a gate is suddenly dropped so that the discharge locally ($x=0$) is instantaneously ($t=0$) changed. (reduced). This will pile up water upstream of the gate and cause an increase in water depth at the gate. This depth increase will move upstream from the gate at a speed "C" with a depth h_1 = original depth of flow ahead of it and a depth $h_2 > h_1$ behind it. This situation corresponds to a moving hydraulic jump (a.k.a. a bore) and is shown below



This is obviously an unsteady problem! But we may make it steady by adopting a coordinate system that moves along with the bore, i.e. moves at a constant speed "C" in the upstream direction. In this moving coordinate system (i) the depth change from h_1 to h_2 is not changing position (we are moving along with it) and (ii) the velocity upstream and downstream of the jump is $C+V_1$ and $C+V_2$, respectively.

Thus, in the moving reference frame the moving hydraulic jump appears identical to the stationary hydraulic jump we just analyzed, except for the addition of "c" to all velocities.

Steady "Bore" in Moving Coordinate System

Conservation of volume: $(C+V_1)h_1 = (C+V_2)h_2$

Conservation of momentum: $(\rho(C+V_1)^2 + \frac{1}{2}\rho g h_1)h_1 =$

$$(\rho(C+V_2)^2 + \tfrac{1}{2}\rho g h_2)h_2$$

Exactly the same as before when V_1 and V_2 are replaced by (V_1+C) and (V_2+C), respectively. For the stationary hydraulic jump we obtained

$$Fr_1^{\,2} \text{ now equal to } \frac{(V_1+C)^2}{g h_1} = \frac{1}{2}\frac{h_2}{h_1}\left(1+\frac{h_2}{h_1}\right)$$

from which C = the speed of the bore, may be obtained if V_1, h_1 and h_2 are known. V_1 and h_1 correspond to the original steady uniform flow in the channel and they may be obtained from knowledge of S_0, "n", b, and Q. If the gate is partially closed the elevation jump h_2 may be estimated by requiring the discharge Q to pass under the gate.

In particular, if we assume the jump to be weak, i.e. $h_2 - h_1 \ll h_1$ so $h_2/h_1 - 1 \ll 1$ or $h_2/h_1 \approx 1$, we obtain from the moving jump condition

$$(V_1 + c)^2 = \frac{1}{2}\frac{h_2}{h_1}\left(1+\frac{h_2}{h_1}\right)gh_1 \approx gh_1$$

or

$$V_1 + c = \pm\sqrt{gh_1} \Rightarrow \underline{c = -V_1 \pm \sqrt{gh_1}}$$

Thus, an infinitessimaly small disturbance on still water ($V_1 = 0$) may propagates in the upstream or downstream direction at a speed

$$C_0 = \begin{cases} +\sqrt{gh_1} = \sqrt{gh} & \text{upstream} \\ -\sqrt{gh_1} = -\sqrt{gh} & \text{downstream} \end{cases}$$

When the small disturbance moves on running water, its speed may be

$$c = -V_1 \begin{cases} +\sqrt{gh_1} \\ -\sqrt{gh_1} \end{cases}$$

<u>If $V_1 < \sqrt{gh_1} \Rightarrow$ i.e. $Fr = \frac{V_1}{\sqrt{gh}} < 1$ Subcritical Flow</u>

then

$$c = \begin{cases} \sqrt{gh_1}\,(1 - Fr) > 0 & \text{moving upstream} \\ -\sqrt{gh_1}\,(1 + Fr) < 0 & \text{moving downstream} \end{cases}$$

Thus, in a subcritical flow disturbances in the flow propagates both in the upstream and downstream directions.

<u>If $V_1 > \sqrt{gh_1}$, $\Rightarrow$ i.e. $Fr = \frac{V_1}{\sqrt{gh_1}} > 1$: Supercritical Flow</u>

then

$$C = \begin{cases} \sqrt{gh_1}\,(1 - Fr) < 0 \\ -\sqrt{gh_1}\,(1 + Fr) < 0 \end{cases} \quad \text{both move downstream}$$

Thus, in a supercritical flow disturbances can propagate only in the downstream direction

These result demonstrate the significance of the Froude Number

$$Fr = \frac{V}{\sqrt{gh_m}} = \frac{\text{Fluid Velocity}}{\text{Speed of small Disturbance}}$$

<u>$Fr < 1$ Subcritical Flow</u>

Changes in flow conditions are felt upstream of the location where changes occur.

<u>Subcritical Flows ($Fr<1$) are controlled by Downstream Conditions</u>

<u>$Fr > 1$ Supercritical Flow</u>

Changes in flow conditions are felt only downstream of location where changes occur. Upstream is entirely unaware of what happens downstream.

<u>Supercritical Flows ($Fr>1$) are controlled by Upstream Conditions</u>