1.050 Engineering Mechanics I

Lecture 25:
Beam elasticity – problem solving technique and examples

Handout

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1.050 – Content overview

I. Dimensional analysis
1. On monsters, mice and mushrooms
2. Similarity relations: Important engineering tools

Lectures 1-3
Sept.

II. Stresses and strength
3. Stresses and equilibrium
4. Strength models (how to design structures, foundations.. against mechanical failure)

Lectures 4-15
Sept./Oct.

III. Deformation and strain
5. How strain gages work?
6. How to measure deformation in a 3D structure/material?

Lectures 16-19
Oct.

IV. Elasticity
7. Elasticity model – link stresses and deformation
8. Variational methods in elasticity

Lectures 20-31
Oct./Nov.

V. How things fail – and how to avoid it
9. Elastic instabilities
10. Plasticity (permanent deformation)
11. Fracture mechanics

Lectures 32-37
Dec.

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1.050 – Content overview

I. Dimensional analysis

II. Stresses and strength

III. Deformation and strain

IV. Elasticity

Lecture 20: Introduction to elasticity (thermodynamics)
Lecture 21: Generalization to 3D continuum elasticity
Lecture 22: Special case: isotropic elasticity
Lecture 23: Applications and examples
Lecture 24: Beam elasticity
Lecture 25: Applications and examples (beam elasticity)
Lecture 26: … cont'd and closure
…

V. How things fail – and how to avoid it

Beam bending elasticity

Governed by this differential equation:

$$\frac{d^4 \xi_z}{dx^4} = \frac{f_z}{EI}$$

Integration provides solution for displacement

Solve integration constants by applying BCs

Note:

E = material parameter (Young's modulus)

I = geometry parameter (property of cross-section)

f_z = distributed shear force (force per unit length)

$f_z = pb_0$ where p_0=pressure, b=thickness of beam in y-direction

4-step procedure to solve beam elasticity problems

- **Step 1**: Write down BCs (stress BCs and displacement BCs), analyze the problem to be solved (read carefully!)
- **Step 2**: Write governing equations for ξ_z, ξ_x ...
- **Step 3**: Solve governing equations (e.g. by integration), results in expression with unknown integration constants
- **Step 4**: Apply BCs (determine integration constants)

Note: Very similar procedure as for 3D isotropic elasticity problems

Difference in governing equations (simpler for beams)

Physical meaning of derivatives of ξ_z

$$\frac{d^4\xi_z}{dx^4} = \frac{f_z}{EI} \qquad \frac{d^4\xi_z}{dx^4} EI = f_z$$ Shear force density

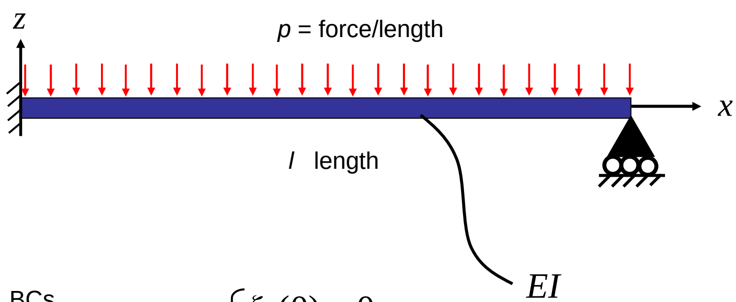
$$\frac{d^3\xi_z}{dx^3} = -\frac{Q_z}{EI} \qquad -\frac{d^3\xi_z}{dx^3} EI = Q_z$$ Shear force

$$\frac{d^2\xi_z}{dx^2} = -\frac{M_y}{EI} \qquad -\frac{d^2\xi_z}{dx^2} EI = M_y$$ Bending moment

$$\frac{d\xi_z}{dx} = -\omega_y \qquad -\frac{d\xi_z}{dx} = \omega_y$$ Rotation (angle)

$$\xi_z \qquad \xi_z$$ Displacement

Step-by-step example



Step 1: BCs

$$x = 0 \quad \begin{cases} \xi_z(0) = 0 \\ \omega_y(0) = 0 \end{cases}$$

$$x = l \quad \begin{cases} \xi_z(l) = 0 \\ M_y(0) = 0 \end{cases}$$

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Step 2: Governing equation

$$\frac{d^4\xi_z}{dx^4} = \frac{f_z}{EI} \longrightarrow \frac{d^4\xi_z}{dx^4} = -\frac{p}{EI}$$

p applied in negative *z*-direction

Step 3: Integration

$$\xi_z'''' = -\frac{p}{EI} \longrightarrow \begin{cases} \xi_z''' = -\frac{p}{EI}x + C_1 \\ \xi_z'' = -\frac{p}{EI}\frac{x^2}{2} + C_1 x + C_2 \\ \xi_z' = -\frac{p}{EI}\frac{x^3}{6} + C_1\frac{x^2}{2} + C_2 x + C_3 \\ \xi_z = -\frac{p}{EI}\frac{x^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4 \end{cases}$$

Step 4: Apply BCs

$$\begin{cases} \xi_z''' = -\dfrac{p}{EI}x + C_1 = -\dfrac{Q_z}{EI} \\ \xi_z'' = -\dfrac{p}{EI}\dfrac{x^2}{2} + C_1 x + C_2 = -\dfrac{M_y}{EI} \\ \xi_z' = -\dfrac{p}{EI}\dfrac{x^3}{6} + C_1\dfrac{x^2}{2} + C_2 x + C_3 = -\omega_y \\ \xi_z = -\dfrac{p}{EI}\dfrac{x^4}{24} + C_1\dfrac{x^3}{6} + C_2\dfrac{x^2}{2} + C_3 x + C_4 \end{cases}$$

Known quantities are marked

Step 4: Apply BCs (cont'd)

$$\xi_z(0) = 0 \rightarrow C_4 = 0$$

$$\omega_y(0) = 0 \rightarrow C_3 = 0$$

$$\begin{cases} \xi_z(l) = 0 \rightarrow -\dfrac{p}{EI}\dfrac{l^4}{24} + C_1\dfrac{l^3}{6} + C_2\dfrac{l^2}{2} = 0 \\ M_y(0) = 0 \rightarrow -\dfrac{p}{EI}\dfrac{l^2}{2} + C_1 l + C_2 = 0 \end{cases}$$

$$\begin{pmatrix} \dfrac{l^3}{6} & \dfrac{l^2}{2} \\ l & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{p}{EI} \begin{pmatrix} \dfrac{l^4}{24} \\ \dfrac{l^2}{2} \end{pmatrix} \longrightarrow \begin{array}{l} C_1 = \dfrac{p}{EI}\dfrac{5}{8}l \\ C_2 = -\dfrac{p}{EI}\dfrac{1}{8}l^2 \end{array}$$

Solution:

$$
\begin{cases}
Q_z(x) = p\left(x - \frac{5}{8}l\right) \\
M_y(x) = p\left(\frac{1}{8}l^2 + \frac{x^2}{2} - \frac{5}{8}lx\right) \\
\omega_y(x) = \frac{p}{EI}\left(\frac{1}{8}l^2 x + \frac{x^3}{6} - \frac{5}{16}lx^2\right) \\
\xi_z(x) = -\frac{p}{EI}\left(\frac{1}{16}l^2 x^2 + \frac{x^4}{24} - \frac{5}{48}lx^3\right)
\end{cases}
$$