Spectral Theorem Example

1.022 Recitation Notes, Paolo Bertolotti

These notes solve for the eigenvalues and eigenvectors of a matrix, discuss their properties briefly, and end with the spectral theorem. Please refer to the Linear Algebra Reference notes for detailed definitions.

Consider the following 2x2 matrix A. A is symmetric, meaning it equals its transpose.

In[1]:=
```
A = {{2, -1}, {-1, 3}};
MatrixForm[A]
```

Out[2]//MatrixForm=

$$\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

In[3]:=
```
MatrixForm[Transpose[A]]
```

Out[3]//MatrixForm=

$$\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

To solve for the eigenvalues of A, we first recall the equation that we used to define eigenvalues and eigenvectors. An eigenvector x, with a corresponding eigenvalue λ (which is a scalar), is a non-zero vector such that
$Ax = \lambda x$

We use the 2x2 identity matrix Id and rewrite the eigenvalue/vector equation to get
$(A - \lambda \, Id) \, x = 0$.

In[9]:=
```
Id = {{1, 0}, {0, 1}};
MatrixForm[Id]
```

Out[10]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
MatrixForm[A - λ * Id]
```

$$\begin{pmatrix} 2-\lambda & -1 \\ -1 & 3-\lambda \end{pmatrix}$$

Since eigenvectors are non-zero, we are looking for a non-zero vector that is in the null-space of (A - λ Id). If (A - λ Id) has a non-zero vector in its null-space, it is not invertible (it is singular), and its determinant is 0. Therefore, we solve for the values of λ that set determinant(A - λ Id) = 0.

For the generic 2x2 matrix F below, its determinant is given by
det(F) = ad - bc.

In[23]:=
```
F = {{a, b}, {c, d}};
MatrixForm[F]
```

Out[24]//MatrixForm=

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Therefore the determinant of (A - λ Id) is:

In[25]:=
```
Simplify[(2 - λ) (3 - λ) - (-1) (-1)]
```

Out[25]= $5 - 5\lambda + \lambda^2$

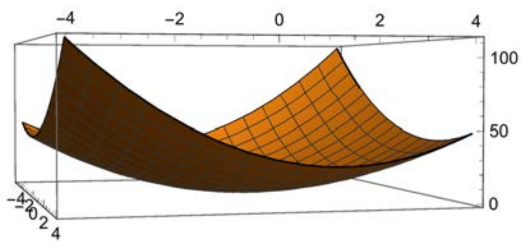
Setting this equation equal to 0, we arrive at the characteristic polynomial of A. Solving this equation provides the eigenvalues of A.

In[26]:=
```
NSolve[5 - 5 λ + λ^2 == 0, λ]
```
Out[26]=
```
{{λ → 1.38197}, {λ → 3.61803}}
```

Since we started with a symmetric matrix, the eigenvalues are real. Note that we have a square, symmetric matrix with non-negative eigenvalues. Therefore, A is positive semi-definite (it is actually positive definite as the eigenvalues are strictly positive). For positive semi-definite matrices, x'Ax ≥ 0 for any vector x. Let's expand out the quadratic form x'Ax and plot it to show that the function is always non-negative, for any value of $x = [x_1, x_2]$.

In[3]:=
```
Expand[Transpose[{{x1}, {x2}}].A.{{x1}, {x2}}]
```
Out[3]= $\{\{2 x_1^2 - 2 x_1 x_2 + 3 x_2^2\}\}$

In[4]:=
```
Plot3D[2 x1^2 - 2 x1 x2 + 3 x2^2, {x1, -4, 4}, {x2, -4, 4}]
```
Out[4]=



Now we turn to solving for the eigenvectors. For each eigenvalue λ_i, we need to solve the system of linear equations $(A - \lambda_i \, Id) \, x = 0$ for x.

In[6]:=
```
λ1 = 1.381966011250105`;
λ2 = 3.618033988749895`;
```

In[11]:=
```
MatrixForm[(A - λ1 * Id)]
```
Out[11]//MatrixForm=

$$\begin{pmatrix} 0.618034 & -1. \\ -1. & 1.61803 \end{pmatrix}$$

For the first eigenvalue, solving the 2x2 system $(A - \lambda_1 \, Id) \, x = 0$ for x below, we see that x is not unique. Any vector pointing in the same direction as the vector below will be an eigenvector of A for λ_1, regardless of its length. Since we care about the direction of the vector, not its length, we impose the additional restriction of unit length (l_2 norm = 1).

In[18]:=
```
NSolve[(A - λ1 * Id).{{x1}, {x2}} == 0, {x1, x2}]
```
Out[18]=
```
{{x1 → 0. + 1.61803 x2}}
```

In[19]:=
```
Solve[Sqrt[(0.` + 1.618033988749895` x2)^2 + x2^2] == 1, x2]
```
Out[19]=
```
{{x2 → -0.525731}, {x2 → 0.525731}}
```

In[23]:=
```
x1 = 0.` + 1.618033988749895` (0.5257311121191336`)
```
Out[23]=
```
0.850651
```

We have our first eigenvector x = [0.850651, 0.525731]. We follow the same steps to solve for the second eigenvector and find y = [-0.525731, 0.850651].

In[25]:=
```
NSolve[(A - λ2 * Id).{{y1}, {y2}} == 0, {y1, y2}]
```
Out[25]= {{$y_1 \to 0. - 0.618034\, y_2$}}

In[26]:=
```
Solve[Sqrt[(0.` - 0.6180339887498948` y2)^2 + y2^2] == 1, y2]
```
Out[26]= {{$y_2 \to -0.850651$}, {$y_2 \to 0.850651$}}

In[27]:=
```
y1 = 0.` - 0.6180339887498948` (0.85065080835204`)
```
Out[27]= −0.525731

Now, since we started with a symmetric matrix, we know the eigenvectors will be orthogonal.

In[28]:=
```
x = {{0.85065080835204`}, {0.5257311121191336`}};
y = {{-0.5257311121191336`}, {0.85065080835204`}};
Transpose[x].y
```
Out[30]= {{0.}}

Putting the eigenvectors into the matrix X with columns x and y, we have an orthogonal matrix. For orthogonal matrices, we know their transpose equals their inverse.

```
X = {{0.85065080835204`, -0.5257311121191336`},
   {0.5257311121191336`, 0.85065080835204`}};

MatrixForm[X]
```
Out[33]//MatrixForm=

$$\begin{pmatrix} 0.850651 & -0.525731 \\ 0.525731 & 0.850651 \end{pmatrix}$$

In[34]:=
```
Transpose[X] == Inverse[X]
```
Out[34]= True

Combining all the eigenvalue/vector equations, we have AX = XΛ where X is our eigenvector matrix (with the eigenvectors as columns) and Λ is a diagonal matrix with the eigenvalues on the diagonal. Multiplying both the right and left sides of the equation on the right by X^{-1}, we get $A = X\Lambda X^{-1}$. As X is orthogonal, we arrive at the spectral theorem
$A = X\Lambda X^T$

In[35]:=
```
Λ = {{λ1, 0}, {0, λ2}};
MatrixForm[Λ]
```
Out[36]//MatrixForm=

$$\begin{pmatrix} 1.38197 & 0 \\ 0 & 3.61803 \end{pmatrix}$$

In[38]:=
```
MatrixForm[X.Λ.Transpose[X]]
```
Out[38]//MatrixForm=

$$\begin{pmatrix} 2. & -1. \\ -1. & 3. \end{pmatrix}$$