THE RELATIONSHIP BETWEEN $U(t,t_0)$ AND $c_n(t)$

For a time-dependent Hamiltonian, we can often partition

$$H = H_0 + V(t)$$

H_0: time-independent; $V(t)$: time-dependent potential. We know the eigenkets and eigenvalues of H_0:

$$H_0|n\rangle = E_n|n\rangle$$

We describe the initial state of the system $(t = t_0)$ as a superposition of these eigenstates:

$$|\psi(t_0)\rangle = \sum_n c_n |n\rangle$$

For longer times t, we would like to describe the evolution of $|\psi\rangle$ in terms of an expansion in these kets:

$$|\psi(t)\rangle = \sum_n c_n(t)|n\rangle$$

The expansion coefficients $c_k(t)$ are given by

$$c_k(t) = \langle k|\psi(t)\rangle = \langle k|U(t,t_0)|\psi(t_0)\rangle$$

Alternatively we can express the expansion coefficients in terms of the interaction picture wavefunctions

$$b_k(t) = \langle k|\psi_I(t)\rangle$$

(This notation follows Cohen-Tannoudji.) Notice

$$\begin{aligned} c_k(t) &= \langle k|\psi(t)\rangle = \langle k|U_0 U_I|\psi(t_0)\rangle \\ &= e^{-i\omega_k t}\langle k|U_I|\psi(t_0)\rangle \\ &= e^{-i\omega_k t}\, b_k(t) \end{aligned}$$

so that $|b_k(t)|^2 = |c_k(t)|^2$. Also, $b_k(0) = c_k(0)$. It is easy to calculate $b_k(t)$ and then add in the extra oscillatory term at the end.

Now, starting with

$$i\hbar \frac{\partial |\psi_I\rangle}{\partial t} = V_I |\psi_I\rangle$$

we can derive an equation of motion for b_k

$$i\hbar \frac{\partial b_k}{\partial t} = \langle k | V_I U_I | \psi_I(t_0)\rangle \qquad \psi_I(t_0) = \sum_n b_n |n\rangle$$

inserting $\sum_n |n\rangle\langle n| = 1$

$$= \sum_n \langle k | V_I | n\rangle \langle n | U_I | \psi_I(t_0)\rangle$$

$$= \sum_n \langle k | V_I | n\rangle \, b_n(t)$$

$$i\hbar \frac{\partial b_k}{\partial t} = \sum_n V_{kn}(t)\, e^{-i\omega_{nk}t}\, b_n(t)$$

This equation is an exact solution. It is a set of coupled differential equations that describe how probability amplitude moves through eigenstates due to a time-dependent potential. Except in simple cases, these equations can't be solved analytically, but it's often straightforward to integrate numerically.

Exact Solution: Resonant Driving of Two-level System

Let's describe what happens when you drive a two-level system with an oscillating potential.

$$V(t) = V\cos\omega t = Vf(t)$$

This is what you expect for an electromagnetic field interacting with charged particles: dipole transitions. The electric field is

$$\overline{E}(t) = \overline{E}_0 \cos\omega t$$

For a particle with charge q in a field $\overline{E}$, the force on the particle is

$$\overline{F} = q\overline{E}$$

which is the gradient of the potential

$$F_x = -\frac{\partial V}{\partial x} = qE_x \quad \Rightarrow \quad V = -qE_x x$$

qx is just the x component of the dipole moment μ. So matrix elements in V look like:

$$\langle k \,|\, V(t) \,|\, \ell \rangle = -qE_x \langle k \,|\, x \,|\, \ell \rangle \cos \omega t$$

More generally,

$$V = -\overline{E} \cdot \overline{\mu} \,.$$

So,

$$V(t) = V \cos \omega t = -\overline{E}_0 \cdot \overline{\mu} \cos \omega t \,.$$

$$V_{k\ell}(t) = V_{k\ell} \cos \omega t = -\overline{E}_0 \cdot \overline{\mu}_{k\ell} \cos \omega t$$

We will now couple our two states $|k\rangle + |\ell\rangle$ with the oscillating field. Let's ask if the system starts in $|\ell\rangle$ what is the probability of finding it in $|k\rangle$ at time t?

The system of differential equations that describe this situation are:

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} b_k(t) &= \sum_n b_n(t) V_{kn}(t)\, e^{-\omega_{nk} t} \\ &= \sum_n b_n(t) V_{kn}\, e^{-i\omega_{nk} t} \times \tfrac{1}{2}\left(e^{-i\omega t} + e^{+i\omega t}\right) \end{aligned}$$

$$i\hbar \dot{b}_k = \tfrac{1}{2} b_\ell V_{k\ell} \left[e^{i(\omega_{k\ell} - \omega)t} + e^{i(\omega_{k\ell} + \omega)t} \right] + \tfrac{1}{2} b_k \cancel{V_{kk}} \left[e^{i\omega t} + e^{-i\omega t} \right] \qquad = (1) \text{ and } (2)$$

$$i\hbar \dot{b}_\ell = \tfrac{1}{2} b_\ell \cancel{V_{\ell\ell}} \left[e^{i\omega t} + e^{-i\omega t} \right] + \tfrac{1}{2} b_k V_{\ell k} \left[e^{i(\omega_{\ell k} - \omega)t} + e^{i(\omega_{\ell k} + \omega)t} \right] \qquad = (3) \text{ and } (4)$$

or

$$\left[e^{-i(\omega_{k\ell} + \omega)t} + e^{-i(\omega_{k\ell} - \omega)t} \right]$$

We can drop (2) and (3). For our case, $V_{ii} = 0$.

We also make the **<u>secular approximation</u>** (rotating wave approximation) in which the nonresonant terms are dropped. When $\omega_{k\ell} \approx \omega$, terms like $e^{\pm i\omega t}$ or $e^{i(\omega_{k\ell} + \omega)t}$ oscillate very rapidly and so don't contribute much to change of c_n.

So we have:

$$\dot{b}_k = \tfrac{-i}{2\hbar} b_\ell \, V_{k\ell} \, e^{i(\omega_{k\ell}-\omega)t} \tag{1}$$

$$\dot{b}_\ell = \tfrac{-i}{2\hbar} b_k \, V_{\ell k} \, e^{-i(\omega_{k\ell}-\omega)t} \tag{2}$$

Note that the coefficients are oscillating out of phase with one another.

Now if we differentiate (1):

$$\ddot{b}_k = \tfrac{-i}{2\hbar}\left[\dot{b}_\ell \, V_{k\ell} \, e^{i(\omega_{k\ell}-\omega)t} + i\left(\omega_{k\ell}-\omega\right) b_\ell \, V_{k\ell} \, e^{i(\omega_{k\ell}-\omega)t}\right] \tag{3}$$

Rewrite (1):

$$b_\ell = \frac{2i\hbar}{V_{k\ell}} \dot{b}_k \, e^{-i(\omega_{k\ell}-\omega)t} \tag{4}$$

and substitute (4) and (2) into (3), we get linear second order equation for b_k.

$$\ddot{b}_k - i\left(\omega_{k\ell}-\omega\right)\dot{b}_k + \frac{\left|V_{k\ell}\right|^2}{4\hbar^2} b_k = 0$$

This is just the second order differential equation for a damped harmonic oscillator:

$$a\ddot{x} + b\dot{x} + cx = 0$$

$$x = e^{-(b/2a)t}\left(A\cos\mu t + B\sin\mu t\right) \quad \mu = \tfrac{1}{2a}\left[4ac - b^2\right]^{1/2}$$

With a little more work, we find (remember $b_k(0)=0$ and $b_\ell(0)=1$)

$$P_k = \left|b_k\left(t\right)\right|^2 = \frac{\left|V_{k\ell}\right|^2}{\left|V_{k\ell}\right|^2 + \hbar^2\left(\omega_{k\ell}-\omega\right)^2} \sin^2 \Omega_r t$$

$$\Omega_R = \tfrac{1}{2\hbar}\left[\left|V_{k\ell}\right|^2 + \hbar^2\left(\omega_{k\ell}-\omega\right)^2\right]^{1/2}$$

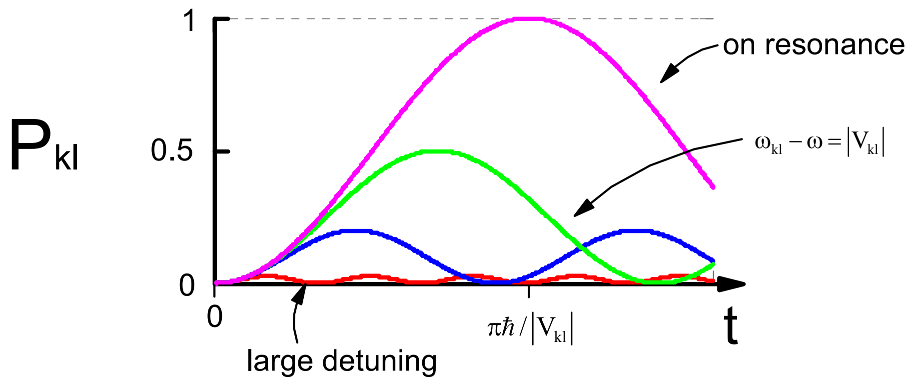
$$P_\ell = 1 - \left|b_k\right|^2$$

Amplitude oscillates back and forth between the two states at a frequency dictated by the coupling.

<u>Resonance</u>: To get transfer of probability amplitude you need the driving field to be at the same frequency as the energy splitting.

Note a result we will return to later: Electric fields couple states, creating coherences!

On resonance, you always drive probability amplitude entirely from one state to another.



Efficiency of driving between ℓ and k states drops off with detuning.