3D-Central Force Problems II. Levi-Civita: ε_{ijk}.

Last time: * $[\mathbf{x},\mathbf{p}] = i\hbar \rightarrow$ use to obtain vector commutation rules: generalize from 1-D to 3-D
* we have conjugate position and momentum components in Cartesian coordinates

Correspondence Principle Recipe
Cartesian coordinates and vector analysis
Symmetrize (make it Hermitian)
classical mechanics in $\hbar \rightarrow 0$ limit

Derived key results:

$$\left[f(\mathbf{x}),\mathbf{p_x}\right] = i\hbar\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \Rightarrow \left[f(\mathbf{r}),\mathbf{p}_x\right] = i\hbar\frac{\partial \mathbf{f}}{\partial \mathbf{r}}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\mathbf{x}} = i\hbar\frac{\partial \mathbf{f}}{\partial \mathbf{r}}\frac{\mathbf{x}}{\mathbf{r}}$$

$$\left[f(\mathbf{r}),\mathbf{q}\cdot\mathbf{p}\right] = i\hbar\frac{\partial \mathbf{f}}{\partial \mathbf{r}}\mathbf{r} \text{ based on } \frac{\partial \mathbf{f}}{\partial \mathbf{r}}\left(\frac{\partial \mathbf{r}}{\partial \mathbf{x}}+\frac{\partial \mathbf{r}}{\partial \mathbf{y}}+\frac{\partial \mathbf{r}}{\partial \mathbf{z}}\right) \text{ and } \mathbf{r} = \left[\mathbf{x}^2+\mathbf{y}^2+\mathbf{z}^2\right]^{1/2}$$

* $\mathbf{p_r} = \mathbf{r}^{-1}\left(\mathbf{q}\cdot\mathbf{p} - i\hbar\right) \leftarrow$ (came from symmetrization in Cartesian coordinates)

* $\mathbf{p}^2 = \mathbf{p_r}^2 + \mathbf{r}^{-2}\mathbf{L}^2 \leftarrow$ separated $\mathbf{p}_{\parallel}$ from $\mathbf{p}_{\perp}$ — *operator* algebra gave simple separation of variables

* $\mathbf{L} = \mathbf{q}\times\mathbf{p}$ — not necessary (or possible) to symmetrize

* $\mathbf{H} = \dfrac{\mathbf{p_r}^2}{2\mu} + \left[\boxed{\dfrac{\mathbf{L}^2}{2\mu\mathbf{r}^2}} + V(\mathbf{r})\right]$ — $V_\ell(r)$ radial effective potential

need to show we can ignore the order of L^2 and r^2

We do not yet know anything about the eigenstates and the eigenvalues and eigenstates of $\mathbf{L}^2$ and $\mathbf{L}_i$.

TODAY [purpose is mostly to practice commutation rule [,], and angular momentum algebras]

* Obtain angular Momentum Commutation Rules $\rightarrow$ Block diagonalize **H**
* ε_{ijk} Levi-Civita Antisymmetric Tensor
useful in derivations, vector commutators, and remembering stuff.

Next Lecture: Begin derivation of all angular momentum matrix elements starting from the Commutation Rule definitions of angular momentum components.

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GOALS

1. $\left[\mathbf{L}_i, f(r)\right] = 0$ any scalar function of scalar r.
2. $\left[\mathbf{L}_i, \mathbf{p_r}\right] = 0$ difficult - need to use ε_{ijk}!
3. $\left[\mathbf{L}_i, \mathbf{p}_\mathbf{r}^2\right] = 0$
4. $\left[\mathbf{L}_i, \mathbf{L}^2\right] = 0$ $\left(\text{but } \left[\mathbf{L}_i, \mathbf{L}_j^2\right] \neq 0!\right)$
5. *C.S.C.O.* $\mathbf{H}, \mathbf{L}^2, \mathbf{L}_i \rightarrow$ enable block diagonalization of **H**

$\mathbf{L}^2$ and $\mathbf{L}_i$ block-diagonalize **H** according to different eigenvalues of $\mathbf{L}^2$ and $\mathbf{L}_i$.
Items 1-4 are chosen to show that all terms in **H** commute with $\mathbf{L}^2$ and $\mathbf{L}_i$

$\mathbf{L}_i$: choose $\mathbf{L}_z$ for example

1. $$\left[\mathbf{L}_z, f(\mathbf{r})\right] = \left[\mathbf{x}\mathbf{p}_y - \mathbf{y}\mathbf{p}_x, f(\mathbf{r})\right] = \mathbf{x}\left[\mathbf{p}_y, f\right] + [\mathbf{x}, f]\mathbf{p}_y - \mathbf{y}\left[\mathbf{p}_x, f\right] - [\mathbf{y}, f]\mathbf{p}_x$$

$$[\mathbf{x}, f] = 0, \qquad [\mathbf{y}, f] = 0 \text{ because } \left[\vec{\mathbf{q}}, f(\mathbf{r})\right] = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\text{recall } \left[f(\mathbf{r}), \mathbf{p}_x\right] = i\hbar\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = i\hbar\frac{\partial f}{\partial r}\frac{x}{r}$$

$$\left[\mathbf{L}_z, f(\mathbf{r})\right] = -i\hbar\frac{\partial f}{\partial r}\left[x\frac{y}{r} - y\frac{x}{r}\right] = 0$$

2. $$\left[\mathbf{L}_z, \mathbf{p_r}\right] = \left[\mathbf{L}_z, \mathbf{r}^{-1}\left(\mathbf{q}\cdot\mathbf{p} - i\hbar\right)\right] = \left[\mathbf{L}_z, \mathbf{r}^{-1}\mathbf{q}\cdot\mathbf{p}\right]$$ (already know that $[\mathbf{L}_z, \mathbf{r}^{-1}i\hbar] = 0$)

$$= \underbrace{\left[\mathbf{L}_z, \mathbf{r}^{-1}\right]}_{0}\mathbf{q}\cdot\mathbf{p} + \mathbf{r}^{-1}\left[\mathbf{L}_z, \mathbf{q}\cdot\mathbf{p}\right]$$

$\frac{1}{r}$ is $f(\mathbf{r})$ and we just showed this commutation rule = 0

$$\left[\mathbf{L}_z, \mathbf{q}\cdot\mathbf{p}\right] = \mathbf{q}\cdot\left[\mathbf{L}_z, \vec{\mathbf{p}}\right] + \left[\mathbf{L}_z, \vec{\mathbf{q}}\right]\cdot\mathbf{p}$$ two vector commutators on RHS

Note that vector $\vec{\mathbf{q}}$ is a not scalar f($\mathbf{r}$)!

need to define special notational trick to evaluate these DIFFICULT COMMUTATORS

Either $A' = A$ or $\mathbf{H}_{AA'} = 0$. If $A = A'$, it is still possible to find linear combination of different eigenstates of **A** (with same-A eigenvalues of **A**) that diagonalizes the associated block of **H**.

$$[\mathbf{H},\mathbf{A}]=0$$
$$0=\langle A|[\mathbf{H},\mathbf{A}]|A'\rangle=\langle A|[-\mathbf{AH}+\mathbf{HA}]|A'\rangle=(-A+A')\mathbf{H}_{AA'}$$
so either $A=A'$ or $\mathbf{H}_{AA'}=0$

Now for something very special and useful

Levi-Civita Symbol	ε_{ijk}
cyclic order	$\varepsilon_{xyz}=\varepsilon_{yzx}=\varepsilon_{zxy}=+1$
adjacent interchange	$\varepsilon_{yxz}=\varepsilon_{zyx}=\varepsilon_{xzy}=-1$ (anti-cyclic order)
2 repeated indices	$\varepsilon_{xxy}=$ etc. $=0$

I claim $[\mathbf{L}_i,\mathbf{p}_j]=i\hbar\sum_k \varepsilon_{ijk}\mathbf{p}_k$. This will become the *definition* of a "vector operator"□ *with respect to* $\mathbf{L}$! [[vector = 1st rank tensor]

Nonlecture: Verify claim for 1 of 3 × 3 = 9 possible cases
let i = x, j = y

Do This!

$$[\mathbf{L}_x,\mathbf{p}_y]=[\mathbf{y}\mathbf{p}_z-\mathbf{z}\mathbf{p}_y,\mathbf{p}_y]=\mathbf{y}[\mathbf{p}_z,\mathbf{p}_y]+[\mathbf{y},\mathbf{p}_y]\mathbf{p}_z-\mathbf{z}[\mathbf{p}_y,\mathbf{p}_y]-[\mathbf{z},\mathbf{p}_y]\mathbf{p}_y$$
$$=0+i\hbar\mathbf{p}_z-0-0$$

Now check this using ε_{ijk}

$$[\mathbf{L}_x,\mathbf{p}_y]=i\hbar\sum_k \varepsilon_{xyk}\mathbf{p}_k=i\hbar[\cancelto{0}{\varepsilon_{xyx}\mathbf{p}_x}+\cancelto{0}{\varepsilon_{xyy}\mathbf{p}_y}+\varepsilon_{xyz}\mathbf{p}_z]$$
$$=i\hbar\mathbf{p}_z. \qquad \text{OK}$$

All other 8 cases go similarly. Feel the power of ε_{ijk}!

Other important Commutation Rules

$$\left.\begin{array}{l}[\mathbf{L}_i,\mathbf{p}_j]=i\hbar\sum_k \varepsilon_{ijk}\mathbf{p}_k \\ [\mathbf{L}_i,\mathbf{q}_j]=i\hbar\sum_k \varepsilon_{ijk}\mathbf{q}_k\end{array}\right\}$$ general definition of a "vector" operator

$\vec{\mathbf{q}}$ and $\vec{\mathbf{p}}$ are examples of vector operators. Classify as vectors *with respect to* $\mathbf{L}$!

$$[\mathbf{L}_i,\mathbf{L}_j]=i\hbar\sum_k \varepsilon_{ijk}\mathbf{L}_k$$ general definition of an "angular momentum." Works even for spin where a $\mathbf{q}\times\mathbf{p}$ definition cannot exist. This is the MOST IMPORTANT STEP

All angular momentum matrix elements will be derived next lecture from these commutation rules.

FOR THE READER: VERIFY <u>ONE COMPONENT</u> OF EACH OF THE THREE ABOVE COMMUTATORS

$\left[\mathbf{L}_i, \mathbf{L}_j\right] = i\hbar \sum_k \varepsilon_{ijk} \mathbf{L}_k$ is identical to

$$\mathbf{L} \times \mathbf{L} = i\hbar \mathbf{L}$$

(expect 0! because vector cross product $\left|\vec{A} \times \vec{B}\right| = |A|\,|B| \sin\theta_{AB}$)

note reversal of x and z terms

$$\mathbf{L} \times \mathbf{L} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \mathbf{L}_x & \mathbf{L}_y & \mathbf{L}_z \\ \mathbf{L}_x & \mathbf{L}_y & \mathbf{L}_z \end{pmatrix} = \hat{i}\left(\mathbf{L}_y\mathbf{L}_z - \mathbf{L}_z\mathbf{L}_y\right) + \hat{j}\left(\mathbf{L}_z\mathbf{L}_x - \mathbf{L}_x\mathbf{L}_z\right) + \hat{k}\left(\mathbf{L}_x\mathbf{L}_y - \mathbf{L}_y\mathbf{L}_x\right)$$

$$= i\hbar\left[\hat{i}\mathbf{L}_x + \hat{j}\mathbf{L}_y + \hat{k}\mathbf{L}_z\right] = i\hbar\mathbf{L}$$

This vector cross product definition of **L** is more general than **q** × **p** because there is no way to define spin in **q** × **p** form but **S** × **S** = $i\hbar$**S** is quite meaningful.

Ask: Can one generalize that, if **L** × **L** = $i\hbar$**L** (instead of 0), and the [$\mathbf{L}_i$, $\mathbf{L}_j$] and [$\mathbf{L}_i$, $\mathbf{p}_j$] commutation rules have similar forms, that **L** × **p** = $i\hbar$**p**? NO! Check for yourself!

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2. Continued. use $\mathbf{p}_r = \mathbf{r}^{-1}\left(\mathbf{q} \cdot \mathbf{p} - i\hbar\right)$

already know that this commutes with $\mathbf{L}_i$

$$\left[\mathbf{L}_z, \mathbf{p}_r\right] = \mathbf{r}^{-1}\vec{\mathbf{q}} \cdot \left[\mathbf{L}_z, \vec{\mathbf{p}}\right] + \mathbf{r}^{-1}\left[\mathbf{L}_z, \vec{\mathbf{q}}\right] \cdot \vec{\mathbf{p}}$$

vector commutators

evaluate the first term

$$\left[\mathbf{L}_i, \vec{\mathbf{p}}\right] = i\hbar \sum_k \left(\hat{i}\varepsilon_{ixk} + \hat{j}\varepsilon_{iyk} + \hat{k}\varepsilon_{izk}\right)\mathbf{p}_k$$

sum of 3 terms

$$\mathbf{q} \cdot \left[\mathbf{L}_i, \vec{\mathbf{p}}\right] = i\hbar \sum_k \left(\mathbf{x}\varepsilon_{ixk} + \mathbf{y}\varepsilon_{iyk} + \mathbf{z}\varepsilon_{izk}\right)\mathbf{p}_k$$

call this an index-j sum

$$= i\hbar \sum_{j,k} \varepsilon_{ijk} \mathbf{q}_j \mathbf{p}_k \qquad (1)$$

only one of these terms is nonzero (but use simpler form)

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and evaluate the second term $[\mathbf{L}_i,\vec{\mathbf{q}}]\cdot\vec{\mathbf{p}}$

$$[\mathbf{L}_i,\vec{\mathbf{q}}]=i\hbar\sum_k\left[\hat{i}\,\varepsilon_{ixk}+\hat{j}\varepsilon_{iyk}+\hat{k}\varepsilon_{izk}\right]\mathbf{q}_k$$

$$[\mathbf{L}_i,\vec{\mathbf{q}}]\cdot\mathbf{p}=i\hbar\sum_k\left[\varepsilon_{ixk}\mathbf{q}_k\mathbf{p}_x+\varepsilon_{iyk}\mathbf{q}_k\mathbf{p}_y+\varepsilon_{izk}\mathbf{q}_k\mathbf{p}_z\right]$$ (a 2nd –index sum)

$$=i\hbar\sum_{j,k}\varepsilon_{ijk}\mathbf{q}_k\mathbf{p}_j=i\hbar\sum_{k,j}\varepsilon_{\mathrm{ikj}}\mathbf{q}_\mathrm{j}\mathbf{p}_\mathrm{k}$$

sum is over j and k, so can permute the k↔j labels

$$=-i\hbar\sum_{k,j}\varepsilon_{\mathrm{ijk}}\mathbf{q}_j\mathbf{p}_k \qquad (2)$$

switch order of j and k

putting Eqs. (1) and (2) together

$$\vec{\mathbf{q}}\cdot[\mathbf{L}_i,\vec{\mathbf{p}}]+[\mathbf{L}_i,\vec{\mathbf{q}}]\cdot\vec{\mathbf{p}}=i\hbar\sum_{j,k}\left[\varepsilon_{ijk}\mathbf{q}_j\mathbf{p}_k-\varepsilon_{ijk}\mathbf{q}_j\mathbf{p}_k\right]=0!$$

The 2 terms from the $[\mathbf{L},\mathbf{p}\cdot\mathbf{q}]$ are combined here.

Elegance and power of $\varepsilon_{\mathrm{ijk}}$ notation!

We have shown, for $\mathbf{p}_r=\mathbf{r}^{-1}(\mathbf{q}\cdot\mathbf{p}-i\hbar)$, that:

* $[\mathbf{L}_\mathrm{i},\mathbf{p}_\mathrm{r}]=0$ for all i
* easy now to show $[\mathbf{L}_\mathrm{i},\mathbf{p}_\mathrm{r}^2]=0$

Finally $$[\mathbf{L}_i,\mathbf{L}^2]=\sum_j[\mathbf{L}_i,\mathbf{L}_j^2]=\sum_j\left(\mathbf{L}_j[\mathbf{L}_i,\mathbf{L}_j]+[\mathbf{L}_i,\mathbf{L}_j]\mathbf{L}_j\right]$$

$$=\sum_j\left[\mathbf{L}_j\left(i\hbar\sum_k\varepsilon_{ijk}\mathbf{L}_k\right)+\left(i\hbar\sum_k\varepsilon_{ijk}\mathbf{L}_k\right)\mathbf{L}_j\right]$$

sum is over j and k, so can permute the j & k indices

same trick: permute j ↔ k indices in second term

Thus $\varepsilon_{\mathrm{ijk}}=-\varepsilon_{ikj}$ $\qquad -\left(i\hbar\sum_k\varepsilon_{ijk}\mathbf{L}_j\right)\mathbf{L}_k$

$$[\mathbf{L}_i,\mathbf{L}^2]=0$$

But be careful: L_i and L_j^2 do not commute even though L_i and L^2 do commute

$$[\mathbf{L}_i,\mathbf{L}_\mathrm{j}^2]=\mathbf{L}_j[\mathbf{L}_i,\mathbf{L}_j]+[\mathbf{L}_i,\mathbf{L}_j]\mathbf{L}_j=i\hbar\left(\mathbf{L}_j\sum_k\varepsilon_{ijk}\mathbf{L}_k+\left(\sum_k\varepsilon_{ijk}\mathbf{L}_k\right)\mathbf{L}_j\right)\neq 0$$

because this is a sum only over k, can't combine and cancel terms. See detail on next page.

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for i=x, j=y

$$\left[\mathbf{L}_x,\mathbf{L}_y^2\right]=\mathbf{L}_y\left[\mathbf{L}_x,\mathbf{L}_y\right]+\left[\mathbf{L}_x,\mathbf{L}_y\right]\mathbf{L}_y=i\hbar\left[\mathbf{L}_y\mathbf{L}_z+\mathbf{L}_z\mathbf{L}_y\right]\neq 0!$$

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so we have shown

$$\left[\mathbf{L}^2,\mathbf{L}_\mathrm{i}\right]=0$$

$$\left[\mathbf{L}^2,\mathrm{f}(\mathbf{r})\right]=0$$

$$\left[\mathbf{L}_\mathrm{i},\mathrm{f}(\mathbf{r})\right]=0$$

$$\left[\mathbf{L}^2,\mathbf{p}_\mathrm{r}\right]=0$$

$$\left[\mathbf{L}_\mathrm{i},\mathbf{p}_\mathrm{r}\right]=0$$

$\therefore$ $\mathbf{L}^2$, $\mathbf{L}_\mathrm{i}$, **H** all commute — **C**omplete **S**et of **M**utually **C**ommuting **O**perators

eigenfunction of $\mathbf{L}^2$ with eigenvalue $\hbar^2\mathrm{L}(\mathrm{L}+1)$

So what does this tell us about $\langle \mathrm{L}|\mathbf{H}|\mathrm{L}'\rangle = ?$ also $\langle M_L|\mathrm{H}|M'_L\rangle$

$L_z|LM_L\rangle=\hbar M_L|LM_L\rangle$

BLOCK DIAGONALIZATION OF **H**!

Both $H_{LL'}=0$ and $H_{M_L,M_L'}=0$

Basis functions $\psi=\chi(\mathrm{r})\left|\mathrm{L}^2,\mathrm{L_z}\right\rangle=\left|\mathrm{nLM_L}\right\rangle$

$\chi(\mathrm{r})$: radial, special
$\left|\mathrm{L}^2,\mathrm{L_z}\right\rangle$: angular, universal
$\mathrm{M_L}$: eigenfunctions of $\mathbf{L}_z$
L: eigenfunctions of $\mathbf{L}^2$
n: which radial eigenfunction?

Next time I will show, starting from

$$\boxed{\left[\mathbf{L}_i,\mathbf{L}_j\right]=i\hbar\sum_k\varepsilon_{ijk}\mathbf{L}_k}$$, that

* $\mathbf{L}^2\left|n\mathrm{L}M_L\right\rangle=\hbar^2\mathrm{L}(\mathrm{L}+1)\left|n\mathrm{L}M_L\right\rangle \qquad \mathrm{L}=0,1,\ldots$

* $\mathbf{L}_z\left|n\mathrm{L}M_L\right\rangle=\hbar M_L\left|n\mathrm{L}M_L\right\rangle \qquad M_L=-L,-L+1,\ldots+L$

also derive all $\mathbf{L}_\mathrm{x}$ and $\mathbf{L}_\mathrm{y}$ matrix elements in $\left|nLM_L\right\rangle$ basis set.

Translation and Rotation Operators

We are interested in QM operators that cause translation or rotation of an initially localized state: $|x_0,y_0,z_0\rangle$ or $|\alpha_0,\beta_0,\gamma_0\rangle$ (where α,β,γ are Euler angles that relate the body-fixed axis system to the laboratory-fixed axis system.

Translations are related to $\hat{p}_x,\hat{p}_y,\hat{p}_z$ operators and rotations are related to $\hat{L}_x,\hat{L}_y,\hat{L}_z$ operators. How do we demonstrate these relationships?

Begin by asking what does $e^{-i\hat{p}_x\delta/\hbar}$ do to an initially localized state $|x_0,y_0, z_0\rangle$.

An initially localized state is an eigenfunction of $\hat{x},\hat{y},\hat{z}$ operators

$$\hat{x}\left|x_0,y_0,z_0\right\rangle = x_0\left|x_0,y_0,z_0\right\rangle$$

similarly for $\hat{y}$ and $\hat{z}$

What does $e^{-i\hat{p}_x\delta/\hbar}$ do to $|x_0,y_0,z_0\rangle$? We want to know to what eigenvalue(s) of $\hat{x}$ does $e^{-i\hat{p}_x\delta/\hbar}\left|x_0,y_0,z_0\right\rangle$ belong? We ask for $\hat{x}\left[e^{-i\hat{p}_x\delta/\hbar}\left|x_0,y_0,z_0\right\rangle\right]$ and we use the commutation rule $\left[\hat{x},f(\hat{p}_x)\right]=i\hbar\dfrac{df}{dp_x}$

$$\hat{x}\left[e^{-i\hat{p}_x\delta/\hbar}\left|x_0,y_0,z_0\right\rangle\right]=\left(e^{-i\hat{p}_x\delta/\hbar}\hat{x}+\left[\hat{x},e^{-i\hat{p}_x\delta/\hbar}\right]\right)\left|x_0,y_0,z_0\right\rangle$$

$$f(\hat{p}_x)=e^{-i\hat{p}_x\delta/\hbar}$$

$$\frac{d\hat{f}}{dp_x}=(-i\delta/\hbar)e^{-i\hat{p}_x\delta/\hbar}$$

$$\left[\hat{x},f(\hat{p}_x)\right]=(i\hbar)(-i\delta/\hbar)e^{-i\hat{p}_x\delta/\hbar}=\delta e^{-i\hat{p}_x\delta/\hbar}$$

Put it all together:

$$\hat{x}\left[e^{-i\hat{p}_x\delta/\hbar}\left|x_0,y_0,z_0\right\rangle\right]=e^{-i\hat{p}_x\delta/\hbar}[x_0+\delta]\left|x_0,y_0,z_0\right\rangle$$

$$\hat{x}\left[e^{-i\hat{p}_x\delta/\hbar}\left|x_0,y_0,z_0\right\rangle\right]=(x_0+\delta)\left[e^{-i\hat{p}_x\delta/\hbar}\left|x_0,y_0,z_0\right\rangle\right]$$

This means that $e^{-i\hat{p}_x\delta/\hbar}\left|x_0,y_0,z_0\right\rangle$ belongs to the $(x_0+\delta)$ eigenvalue of $\hat{x}$!

$$e^{-i\hat{p}_x\delta/\hbar}\left|x_0,y_0,z_0\right\rangle=\left|x_0+\delta,y_0,z_0\right\rangle$$

So we know how to build an operator that causes translations of a localized state in the x, y, or z directions: $\hat{T}_x,\hat{T}_y,\hat{T}_z$.

But we know that $[\hat{p}_i,\hat{p}_j]=0$ for all components of linear momentum. This means that for all linear translations, $[\hat{T}_i,\hat{T}_j]=0$. *The sequence of the linear translations does not matter!* What about rotations of the initially localized state$|\alpha_0,\beta_0,\gamma_0\rangle$? What does $e^{-i\phi\hat{L}_z/\hbar}$ do to $|\alpha_0,\beta_0,\gamma_0\rangle$?

Consider

$$\hat{\alpha}\left[e^{-i\phi\hat{L}_z/\hbar}\left|\alpha_0,\beta_0,\gamma_0\right\rangle\right].$$

Following an argument similar to that for the translational operators

$$\hat{\alpha}e^{-i\phi\hat{L}_z/\hbar}\left|\alpha_0,\beta_0,\gamma_0\right\rangle=\left(\alpha_0+\phi\right)e^{-i\phi\hat{L}_z/\hbar}\left|\alpha_0,\beta_0,\gamma_0\right\rangle$$

$\hat{\alpha}e^{-i\phi\hat{L}_z/\hbar}\left|\alpha_0,\beta_0,\gamma_0\right\rangle$ belongs to the $\alpha_0+\phi$ eigenvalue of $\hat{\alpha}$.

Now show something beautiful: that infinitesimal rotations about different axes do not commute!

$$e^{-i\phi\hat{L}_z/\hbar}e^{-i\theta\hat{L}_y/\hbar}\left|\alpha_0,\beta_0,\gamma_0\right\rangle=\left(e^{-i\theta\hat{L}_y/\hbar}e^{-i\phi\hat{L}_z/\hbar}+\left[e^{-i\phi\hat{L}_z/\hbar},e^{-i\theta\hat{L}_y/\hbar}\right]\right)\left|\alpha_0,\beta_0,\gamma_0\right\rangle$$

Expand the exponentials for infinitesimal θ,ϕ: 1st two terms in the power series expansion of $e^{-i\alpha}$: $1 - i\alpha$.

$$\left[e^{-i\theta\hat{L}_y/\hbar}, e^{-i\phi\hat{L}_z/\hbar}\right] = [1,1] - \left[1, -i\phi\hat{L}_z/\hbar\right] + \left[-i\theta\hat{L}_y/\hbar, 1\right] - \left(-\frac{i}{\hbar}\right)\left[\phi\hat{L}_z, \theta\hat{L}_y\right]$$

$$= 0 + 0 + 0 + \left(-\frac{i}{\hbar}\right)(\phi\theta) i\hbar\hat{L}_x$$

Reversing the order of the rotations about the *y* and *z* axes results in a *non-zero* rotation by $\theta\phi$ about the x-axis!