End of Matrix Solution of H-O, and Feel the Power of the a and a† Operators

1. starting from $\mathbf{H}=\frac{\mathbf{p}^2}{2m}+\frac{1}{2}k\mathbf{x}^2 \quad and\ [\mathbf{x},\mathbf{p}]=i\hbar$

2. we showed $p_{nm}=\frac{m}{i\hbar}x_{nm}\left(E_m-E_n\right)$

$$x_{nm}=\frac{i}{\hbar k}p_{nm}\left(E_m-E_n\right)$$

$$\therefore x_{nn}=0, p_{nn}=0\ and \left\{\begin{array}{c} x_{nm} \\ p_{nm} \end{array}\right\}=0\ if\ E_n=E_m$$

3. $x_{nm}^2=-\frac{1}{km}p_{nm}^2$

$E_m-E_n=\pm\hbar\omega \qquad \omega=\left(k/m\right)^{1/2}$

$\therefore$ the only non-zero **x** and **p** elements are between states whose E' s differ by $\pm\hbar\omega$

4. combs of connected states, "block diagonalization" of **H**, **x**, **p**, $\mathbf{x}^2$, $\mathbf{p}^2$: $E_n^{(i)}=\hbar\omega n+\varepsilon_i$ ← constant offset for block i

5. lowest index must exist because lowest E must exist. Call this index 0

$$\left|x_{01}\right|^2=\frac{\hbar}{2}(km)^{-1/2}$$

$$\left|p_{01}\right|^2=\frac{\hbar}{2}(km)^{+1/2}$$

from arbitrary (but almost universally chosen) phase choice

$$x_{01}=+i(km)^{-1/2}p_{01}$$

Today

6. Recursion Relationship: $\left|x_{nn+1}\right|^2$ in terms of $\left|x_{nn-1}\right|^2$

$\Downarrow$

general matrix elements $\left|x_{nn+1}\right|^2, \left|p_{nn+1}\right|^2$

7. general **x** and **p** elements

8. the only blocks of **H** correspond to $\varepsilon_i=\frac{1}{2}\hbar\omega$

We are ready to derive a powerful, compact, and intuitive algebra:

Dimensionless $\underset{\sim}{\mathbf{x}}$, $\underset{\sim}{\mathbf{p}}$, $\underset{\sim}{\mathbf{H}}$ and **a** (annihilation) and $\mathbf{a}^\dagger$ (creation) operators.

Partial repetition from Lecture #12

phase ambiguity: we can specify absolute phase of **x** or **p** BUT NOT BOTH because that would affect the value of [**x**,**p**]

BY CONVENTION:

matrix elements of **x** are REAL
p are IMAGINARY

try $x_{01} = +i(km)^{-1/2}\, p_{01}$ and plug this into

$$x_{01}p_{01}^{*} - p_{01}x_{01}^{*} = i\hbar$$

get
$$|x_{01}|^2 = \frac{\hbar}{2}(km)^{-1/2}$$
$$|p_{01}|^2 = \frac{\hbar}{2}(km)^{+1/2}$$

[If we had chosen $x_{01} = -i(km)^{-1/2}\, p_{01}$ we would have obtained $|x_{01}|^2 = -\frac{\hbar}{2}(km)^{1/2}$ which is impossible!]

check for self-consistency of seemingly arbitrary phase choices at every opportunity:
* Hermiticity $\left(\mathbf{A}^{\dagger} = \mathbf{A}, \quad A_{ij}^{*} = A_{ji}\right)$
* $|\quad\quad|^2 \geq 0$ for any A_{ij}

6. <u>Recursion Relation for $|x_{ii+1}|^2$</u>

start again with general equation derived in part #3 of Lecture #12 using the phase choice that worked

$$x_{nn+1} = i(km)^{-1/2}\, p_{nn+1}$$

index increasing

Hermiticity
$$x_{n+1n}^{*} = i(km)^{-1/2}\, p_{n+1n}^{*}$$

c.c. of both sides
$$x_{n+1n} = -i(km)^{-1/2}\, p_{n+1n}$$

index decreasing

$$\therefore \quad x_{nn\pm1} = \pm i\left(km\right)^{-1/2} p_{nn\pm1}$$

now the arbitrary part of the phase ambiguity in the relationship between **x** and **p** is eliminated

Apply this to the general term in [**x**,**p**] ⇒ algebra

NONLECTURE : from four terms in [**x**,**p**] = $i\hbar$

$$x_{nn+1}p_{n+1n} = x_{nn+1}p^*_{nn+1} = x_{nn+1}\left(-\frac{(km)^{1/2}}{i}x^*_{nn+1}\right)$$
$$= \left|x_{nn+1}\right|^2\left(+i(km)^{1/2}\right)$$

$$-p_{nn+1}x_{n+1n} = -\left(\frac{(km)^{1/2}}{i}x_{nn+1}\right)\left(x^*_{nn+1}\right) = \left|x_{nn+1}\right|^2\left(+i(km)^{1/2}\right)$$

$$x_{nn-1}p_{n-1n} = x_{nn-1}p^*_{nn-1} = x_{nn-1}\left(+\frac{(km)^{1/2}}{i}x^*_{nn-1}\right)$$
$$= \left|x_{nn-1}\right|^2\left(-i(km)^{1/2}\right)$$

$$-p_{nn-1}x_{n-1n} = -\left(-\frac{(km)^{1/2}}{i}x_{nn-1}\right)\left(x^*_{nn-1}\right) = \left|x_{nn-1}\right|^2\left(-i(km)^{1/2}\right)$$

$$\therefore i\hbar = 2i\left(km\right)^{1/2}\left[\left|x_{nn+1}\right|^2 - \left|x_{nn-1}\right|^2\right]$$

$$\left|x_{nn+1}\right|^2 = \frac{\hbar(km)^{-1/2}}{2} + \left|x_{nn-1}\right|^2$$ recursion relation

but $\left|x_{01}\right|^2 = \left|x_{10}\right|^2 = \frac{\hbar}{2}\left(km\right)^{-1/2}$

thus

$$\boxed{\begin{aligned}\left|x_{nn+1}\right|^2 &= (n+1)\frac{\hbar}{2}\left(km\right)^{-1/2}\\ \left|p_{nn+1}\right|^2 &= (n+1)\frac{\hbar}{2}\left(km\right)^{+1/2}\end{aligned}}$$

general result

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7. Magnitudes and Phases for $x_{nn\pm 1}$ and $p_{nn\pm 1}$

verify phase consistency and Hermiticity for **x** and **p**

in #3 we derived $x_{nn\pm 1} = \pm i(km)^{-1/2} p_{nn\pm 1}$

one self-consistent set is

matrix elements of **x** real and positive

$$x_{nn+1} = +(n+1)^{1/2}\left(\frac{\hbar}{2(km)^{1/2}}\right)^{1/2} = +x_{n+1n}$$

$$x_{nn-1} = +(n)^{1/2}\left(\frac{\hbar}{2(km)^{1/2}}\right)^{1/2} = +x_{nn-1}$$

$$(km)^{1/2} = m\omega$$

AND

matrix elements of **p** imaginary with sign flip for up vs. down

$$p_{nn+1} = -i(n+1)^{1/2}\left(\frac{\hbar(km)^{1/2}}{2}\right)^{1/2} = -p_{n+1n}$$

$$p_{nn-1} = +i(n)^{1/2}\left(\frac{\hbar(km)^{1/2}}{2}\right)^{1/2} = -p_{n-1n}$$

This is the usual phase convention

Phase is a recurrent problem in matrix mechanics because we never look at wavefunctions or evaluate integrals explicitly.

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8. Possible existence of noncommunicating blocks along diagonal of **H, x, p**

you show that $H_{nm} = (n+1/2)\hbar\left(\frac{k}{m}\right)^{1/2}\delta_{nm}$

$\left(\begin{array}{l}\text{note that } \mathbf{x}^2 \text{ and } \mathbf{p}^2 \text{ have non-zero } \Delta n=\pm 2 \text{ elements but} \\ \frac{1}{2}k\mathbf{x}^2 + \frac{\mathbf{p}^2}{2m} \text{ has cancelling contributions in } \Delta n=\pm 2 \text{ locations}\end{array}\right)$

This result implies

* all of the possibly independent blocks in **x**, **p**, **H** are identical
* $\varepsilon_i = (1/2)\hbar\omega$ for all i
* degeneracy of all E_n? all are the same, but can't prove that this universal degeneracy is 1.

End of repetition from Lecture #12

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Creation and Annihilation Operators (CTDL pages 488-508)

* Dimensionless operators
* simple operator algebra rather than complicated real algebra
* matrices arranged according to "selection rules"
* matrix elements calculated by extremely simple rules
* automatic generation of any basis function by repeated operations on lowest (nodeless) basis state

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Get rid of system-specific factors k, μ, ω and also $\hbar$.

$$\omega = (k/m)^{1/2}$$

$$\underset{\sim}{\mathbf{x}} \equiv \left(\frac{m\omega}{\hbar}\right)^{1/2} \mathbf{x}$$

dimensionless ($\underset{\sim}{\mathbf{x}}$), regular ($\mathbf{x}$)

$$\underset{\sim}{\mathbf{p}} \equiv (\hbar m\omega)^{-1/2} \mathbf{p}$$

We choose these factors to make everything come out dimensionless.

$$\underset{\sim}{\mathbf{H}} = \frac{1}{\hbar\omega}\mathbf{H} = \frac{1}{2}\left(\underset{\sim}{\mathbf{x}}^2 + \underset{\sim}{\mathbf{p}}^2\right)$$

$$\mathbf{x}^2 = \left(\frac{\hbar}{m\omega}\right)\underset{\sim}{\mathbf{x}}^2$$

$$\mathbf{p}^2 = \hbar m\omega\, \underset{\sim}{\mathbf{p}}^2$$

$$\mathbf{H} = \frac{1}{2}k\mathbf{x}^2 + \frac{\mathbf{p}^2}{2m} = \frac{1}{2}\hbar\omega\left(\underset{\sim}{\mathbf{x}}^2 + \underset{\sim}{\mathbf{p}}^2\right)$$

$$\boxed{\begin{array}{l}\frac{1}{2}k\left(\frac{\hbar}{m\omega}\right) = \frac{1}{2}\frac{k}{m}\frac{\hbar}{\omega} = \frac{\omega^2}{2}\frac{\hbar}{\omega} = \frac{\hbar\omega}{2} \\ \frac{1}{2m}(\hbar m\omega) = \frac{1}{2}\hbar\omega\end{array}}$$

$$\left[\underset{\sim}{\mathbf{x}}, \underset{\sim}{\mathbf{p}}\right] = \left(\frac{m\omega}{\hbar}\frac{1}{\hbar m\omega}\right)^{1/2}\left[\mathbf{x}, \mathbf{p}\right] = \frac{1}{\hbar}(i\hbar) = i$$

dimensionless

from results for **x, p, H**

$$\underset{\sim}{x}_{mn} = 2^{-1/2}\left[(n+1)^{1/2}\,\delta_{mn+1} + n^{1/2}\delta_{mn-1}\right]$$

$$\underset{\sim}{p}_{mn} = 2^{-1/2}\, i\left[(n+1)^{1/2}\,\delta_{mn+1} - n^{1/2}\delta_{mn-1}\right]$$

$$\underset{\sim}{H}_{mn} = (n+1/2)\,\delta_{mn}$$

diagonal

square root of larger quantum number

note the negative sign

Kronecker − δ's specify selection rules for all nonzero matrix elements

Now define something new: use **a**, $\mathbf{a}^\dagger$ to clean things up even more!

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$$\mathbf{a} = 2^{-1/2}\left(\underset{\sim}{\mathbf{x}} + i\underset{\sim}{\mathbf{p}}\right) \qquad \underset{\sim}{\mathbf{x}} = 2^{-1/2}\left(\mathbf{a} + \mathbf{a}^\dagger\right)$$

$$\mathbf{a}^\dagger = 2^{-1/2}\left(\underset{\sim}{\mathbf{x}} - i\underset{\sim}{\mathbf{p}}\right) \qquad \underset{\sim}{\mathbf{p}} = 2^{-1/2} i\left(\mathbf{a}^\dagger - \mathbf{a}\right)$$

Let's examine the matrix elements of **a** and $\mathbf{a}^\dagger$

now plug in m,n matrix elements of $\underset{\sim}{\mathbf{x}}$ and $\underset{\sim}{\mathbf{p}}$ from previous page

$$a_{mn} = \left[2^{-1/2}\underset{\sim}{x}_{mn} + 2^{-1/2} i\,\underset{\sim}{p}_{mn}\right]$$

$$= \left[\underbrace{\frac{1}{2}(n+1)^{1/2}\delta_{mn+1}}_{\underset{\sim}{\mathbf{x}}} - \underbrace{\frac{1}{2}(n+1)^{1/2}\delta_{mn+1}}_{i\underset{\sim}{\mathbf{p}}} + \underbrace{\frac{1}{2}n^{1/2}\delta_{mn-1}}_{\underset{\sim}{\mathbf{x}}} + \underbrace{\frac{1}{2}n^{1/2}\delta_{mn-1}}_{i\underset{\sim}{\mathbf{p}}}\right]$$

group terms according to "selection rule"

two terms cancel   two terms add

$$\boxed{a_{mn} = n^{1/2}\delta_{mn-1}}$$

the first (left) index is <u>one smaller</u> than the second (right)

column

$$a_{mn} = \langle m|\mathbf{a}|n\rangle = n^{1/2}$$

row

$n^{1/2}\,|\,n-1\rangle$

a is lowering or "annihilation" operator

similarly

$$\boxed{a^\dagger_{mn} = (n+1)^{1/2}\delta_{mn+1}}$$

$a^\dagger_{mn} = \langle m|\mathbf{a}^\dagger|n\rangle = (n+1)^{1/2}$

the first index is one larger than the second
$\mathbf{a}^\dagger$ is a "creation" operator

$$\mathbf{a}^\dagger = \begin{array}{r|c} & \text{columns} \\ & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \text{rows} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1^{1/2} & 0 & 0 & 0 & 0 \\ 0 & 2^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 3^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 4^{1/2} & 0 \end{pmatrix} \end{array}$$

Square root of integers always only one step below main diagonal. **a**, **a**† are obviously not Hermitian!

$$\mathbf{a} = \begin{pmatrix} 0 & 1^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 2^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 3^{1/2} & 0 \\ 0 & 0 & 0 & 0 & 4^{1/2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Square root of integers always only one step above main diagonal.

e.g. $\langle 3|\mathbf{a}^\dagger|2\rangle = 3^{1/2}$

row column

$\mathbf{a}^\dagger$ raises

e.g. $\langle 3|\mathbf{a}|4\rangle = 4^{1/2}$

$\mathbf{a}$ lowers

What is so great about **a**, **a**†?

$\mathbf{a}|n\rangle = n^{1/2}|n-1\rangle$ annihilates 1 quantum

$\mathbf{a}^\dagger|n\rangle = (n+1)^{1/2}|n+1\rangle$ creates 1 quantum

$|n\rangle = [n!]^{-1/2}(\mathbf{a}^\dagger)^n|0\rangle$ generate any state, $|n\rangle$, from the lowest state, $|0\rangle$

needed to normalize. Each application of **a**† gives the next larger integer. Do it n times on $|0\rangle$, get $n!\ |n\rangle$.

More tricks: look at $\mathbf{aa}^\dagger$ and $\mathbf{a}^\dagger\mathbf{a}$

is $\mathbf{aa}^\dagger$ Hermitian? [$(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$] <u>definition of Hermitian</u>

$$\left(\mathbf{aa}^\dagger\right)^\dagger = \mathbf{a}^{\dagger\dagger}\mathbf{a}^\dagger = \mathbf{aa}^\dagger$$

∴ $\mathbf{aa}^\dagger$ and $\mathbf{a}^\dagger\mathbf{a}$ are Hermitian — to what "observable" quantity do they correspond? We will see that one of these is called the "number operator."

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$$\mathbf{aa}^{\dagger} = \frac{1}{2}\left(\underset{\sim}{\mathbf{x}} + i\underset{\sim}{\mathbf{p}}\right)\left(\underset{\sim}{\mathbf{x}} - i\underset{\sim}{\mathbf{p}}\right) = \frac{1}{2}\left(\underset{\sim}{\mathbf{x}}^2 + i\underset{\sim}{\mathbf{p}}\underset{\sim}{\mathbf{x}} - i\underset{\sim}{\mathbf{x}}\underset{\sim}{\mathbf{p}} + \underset{\sim}{\mathbf{p}}^2\right)$$

$$= \frac{1}{2}\left(\underset{\sim}{\mathbf{x}}^2 + \underset{\sim}{\mathbf{p}}^2 - i\underbrace{[\underset{\sim}{\mathbf{x}}, \underset{\sim}{\mathbf{p}}]}_{i}\right) = \frac{1}{2}\left(\underset{\sim}{\mathbf{x}}^2 + \underset{\sim}{\mathbf{p}}^2 + 1\right)$$

similarly $\mathbf{a}^{\dagger}\mathbf{a} = \frac{1}{2}\left(\underset{\sim}{\mathbf{x}}^2 + \underset{\sim}{\mathbf{p}}^2 - 1\right)$

$$\left.\begin{aligned} &\therefore \underset{\sim}{\mathbf{H}} = \frac{1}{2}\left(\mathbf{a}^{\dagger}\mathbf{a} + \mathbf{aa}^{\dagger}\right) \quad \text{and} \quad \left[\mathbf{a}, \mathbf{a}^{\dagger}\right] = 1 \\ &\underset{\sim}{\mathbf{H}} = \mathbf{a}^{\dagger}\mathbf{a} + 1/2 \end{aligned}\right\} \text{simple form for } \underset{\sim}{\mathbf{H}}$$

$\underset{\sim}{\mathbf{H}}$ is the **number operator** + 1/2

$$\mathbf{H} = \hbar\omega\underset{\sim}{\mathbf{H}} = \hbar\omega\left(\mathbf{a}^{\dagger}\mathbf{a} + 1/2\right)$$

# of quanta

$$\mathbf{a}^{\dagger}\mathbf{a}|n\rangle = n|n\rangle \qquad \mathbf{a}^{\dagger}\mathbf{a} \text{ is "number operator"}$$

$$\left[\mathbf{aa}^{\dagger}|n\rangle = (n+1)|n\rangle\right] \quad \text{not as useful}$$

What have we done? We have exposed all of the “symmetry” and universality of the H–O basis set. We can now trivially work out what the matrix for any $\mathbf{x}^n\mathbf{p}^m$ operator looks like and organize it according to selection rules.

What about $\mathbf{x}^3$?

$$\mathbf{x}^3 = \left(\frac{m\omega}{\hbar}\right)^{-3/2} \underset{\sim}{\mathbf{x}}^3$$

When you multiply this out, preserve the order of **a** and $\mathbf{a}^\dagger$ factors.

$$\underset{\sim}{\mathbf{x}}^3 = \left(2^{-3/2}\right)\left(\mathbf{a}+\mathbf{a}^\dagger\right)^3$$

$$= \left(2^{-3/2}\right)\left[\mathbf{a}^3 + \left(\mathbf{a}^\dagger\mathbf{a}\mathbf{a} + \mathbf{a}\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}\mathbf{a}\mathbf{a}^\dagger\right) + \left(\mathbf{a}\mathbf{a}^\dagger\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a}\right) + \mathbf{a}^{\dagger 3}\right]$$

$\Delta n =$ -3, -1, $+1$, $+3$ (# of † minus # of non-†)

Simplify each group using commutation properties so that it has form

$$\mathbf{a}\left[\mathbf{a}^\dagger\mathbf{a}\right]|n\rangle \quad or \quad \mathbf{a}^\dagger\left[\mathbf{a}^\dagger\mathbf{a}\right]|n\rangle$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$n^{1/2} n|n-1\rangle \qquad (n+1)^{1/2} n|n+1\rangle$$

NONLECTURE: Simplify the $\Delta n = -1$ terms.

$$\mathbf{a}^\dagger\mathbf{a}\mathbf{a} = \mathbf{a}\mathbf{a}^\dagger\mathbf{a}\underbrace{-\mathbf{a}\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}^\dagger\mathbf{a}\mathbf{a}}_{[\mathbf{a}^\dagger,\mathbf{a}]\mathbf{a} = -\mathbf{a}} = \mathbf{a}\mathbf{a}^\dagger\mathbf{a} - \mathbf{a}$$

add and subtract the term needed to reverse order

$$\mathbf{a}\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}\mathbf{a}^\dagger\mathbf{a}\underbrace{-\mathbf{a}\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}\mathbf{a}\mathbf{a}^\dagger}_{\mathbf{a}[\mathbf{a},\mathbf{a}^\dagger]=\mathbf{a}} = \mathbf{a}\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}$$

$$\left[\mathbf{a}^\dagger\mathbf{a}\mathbf{a} + \mathbf{a}\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}\mathbf{a}\mathbf{a}^\dagger\right] = 3\mathbf{a}\mathbf{a}^\dagger\mathbf{a}$$

try to put everthing into $\mathbf{a}\mathbf{a}^\dagger\mathbf{a}$ order

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NONLECTURE: Simplify the $\Delta n = +1$ terms.

$$\mathbf{a}\mathbf{a}^\dagger\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger\underbrace{-\mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger + \mathbf{a}\mathbf{a}^\dagger\mathbf{a}^\dagger}_{\left[\mathbf{a},\mathbf{a}^\dagger\right]\mathbf{a}^\dagger=\mathbf{a}^\dagger} = \mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger + \mathbf{a}^\dagger$$

$$\mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a}\underbrace{-\mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger}_{\mathbf{a}^\dagger\left[\mathbf{a},\mathbf{a}^\dagger\right]=\mathbf{a}^\dagger} = \mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}^\dagger$$

$$\left[\mathbf{a}\mathbf{a}^\dagger\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a}\right] = 3\mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a} + 3\mathbf{a}^\dagger$$

$\underline{\Delta n = \pm 3}$

$$\mathbf{a}^3|n\rangle = [n(n-1)(n-2)]^{1/2}|n-3\rangle$$
$$\mathbf{a}^{\dagger 3}|n\rangle = [(n+1)(n+2)(n+3)]^{1/2}|n+3\rangle$$

$\underline{\Delta n = \pm 1}$

$$[\mathbf{a}^\dagger \mathbf{aa} + \mathbf{aa}^\dagger\mathbf{a} + \mathbf{aaa}^\dagger]|n\rangle = 3(n^{3/2})|n-1\rangle$$
$$[\mathbf{aa}^\dagger\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{aa}^\dagger + \mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a}]|n\rangle = 3\left[(n+1)^{1/2} n|n+1\rangle + (n+1)^{1/2}|n+1\rangle\right]$$
$$= 3(n+1)^{1/2}(n+1)|n+1\rangle$$
$$= 3(n+1)^{3/2}|n+1\rangle$$

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no need to do matrix multiplication. Just play with $\mathbf{a}$, $\mathbf{a}^\dagger$ and the $[\mathbf{a}, \mathbf{a}^\dagger]$ commutation rule and the $\mathbf{a}^\dagger\mathbf{a}$ number operator

"Second Quantization"

$$\mathbf{x}^3_{mn} = \left(\frac{\hbar}{2m\omega}\right)^{3/2}\Big[\underbrace{\delta_{mn+3}}_{\text{same as } |n+3\rangle\langle n|}((n+1)(n+2)(n+3))^{1/2}$$
$$+\,\delta_{mn+1}\,3(n+1)^{3/2} \quad \longleftarrow |n+1\rangle\langle n|$$
$$+\,\delta_{mn-1}\,3n^{3/2} \quad \longleftarrow |n\rangle\langle n-1|$$
$$+\,\underbrace{\delta_{mn-3}}_{|n\rangle\langle n-3|}(n(n-1)(n-2))^{1/2}\Big]$$

same as $|n\rangle\langle n-3|$

simple! $\mathbf{x}^3$ is arranged into four separate terms, each with its own explicit selection rule.

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* $V(x) = \frac{1}{2}k\mathbf{x}^2 + \underbrace{a\mathbf{x}^3 + b\mathbf{x}^4}_{\text{anharmonic terms} \rightarrow \text{perturbation theory}}$

* IR transition intensities $\propto |\langle n|\mathbf{x}|n+1\rangle|^2$

* Survival and transfer probabilities of initially prepared pure harmonic oscillator non-eigenstate in an anharmonic potential.

* Expectation values of any function of $\mathbf{x}$ and $\mathbf{p}$.

Universality: all k,m (system-specific) constants are removed until we put them back in at the end of the calculation.

$$\text{e.g., What is } \langle \Delta x \rangle^2 = \left[\langle x^2 \rangle - \langle x \rangle^2 \right]$$

$$\mathbf{x}^2 = \frac{\hbar}{m\omega} \underset{\sim}{x} = \frac{\hbar}{m\omega} \left[\frac{1}{2} \left(\mathbf{a} + \mathbf{a}^\dagger \right)^2 \right] \leftarrow \boxed{\text{pure numbers in [ ]}}$$

$$\langle \Delta x \rangle^2 = \frac{\hbar}{2m\omega} \left[\left\langle \left(\mathbf{a} + \mathbf{a}^\dagger \right)^2 \right\rangle - \left\langle \mathbf{a} + \mathbf{a}^\dagger \right\rangle^2 \right] ?$$

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