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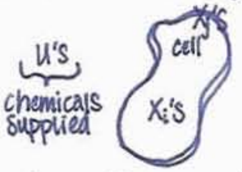
LECTURE 15: NONLINEAR DYNAMICS & STABILITY

THURSDAY
6 APRIL 2006

COMPUTING STEADY-STATES

$$\frac{d}{dt}\vec{x} = A^{(1)}\vec{x} + A^{(2)}\vec{x}\otimes\vec{x} + B^{(1)}\underset{\uparrow \text{ known}}{\vec{u}} + \underbrace{B^{(2)}\vec{u}\otimes\vec{x} + B^{(3)}\vec{u}\otimes\vec{u}}_{\text{ignore for simplicity}}$$



KROENECKER PRODUCT:

$$n \updownarrow \overset{\overset{p}{\leftrightarrow}}{A} \otimes m \updownarrow \overset{\overset{q}{\leftrightarrow}}{B} = \overset{\longleftarrow pq \longrightarrow}{\begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{bmatrix}} \updownarrow nm$$

$$\vec{x}\otimes\vec{x} = \begin{bmatrix} x_1\vec{x} \\ x_2\vec{x} \\ \vdots \\ x_n\vec{x} \end{bmatrix} = \begin{bmatrix} x_1x_1 \\ x_1x_2 \\ \vdots \\ x_1x_n \\ x_2x_1 \\ \vdots \\ x_nx_n \end{bmatrix}$$

REACTIONS ARE IN BALANCE: $X_1 + X_2 \underset{K_r}{\overset{K_f}{\rightleftarrows}} X_3$

In steady-state: $\frac{d}{dt}\underset{\uparrow \text{concentration}}{[x_i]} = 0$

$$\frac{d}{dt}\vec{x} = 0 \quad \left(\begin{smallmatrix}\text{vector}\\ \text{case}\end{smallmatrix}\right)$$

FIND $\vec{x}$'S S.T. $\rightarrow A^{(1)}\vec{x} + A^{(2)}\vec{x}\otimes\vec{x} + B^{1}\vec{u} = 0$

IS THE STEADY-STATE STABLE?

Suppose $\vec{x}^*$: $A^{(1)}\vec{x}^* + A^{(2)}\vec{x}^*\otimes\vec{x}^* + B^{(1)}\vec{u} = 0$

If $\vec{x}(0) = \vec{x}^* + \vec{\epsilon}$ for small $\vec{\epsilon}$,
does soln $\rightarrow \vec{x}^*$?

$$\frac{d}{dt}\vec{x} = A^{(1)}\vec{x} + A^{(2)}\vec{x}\otimes\vec{x} + B^{(1)}\vec{u}$$

only STABLE steady-states are observable
⇓
stable means $\vec{x}^*$ is approached when starting from $\vec{x}^* + \vec{\epsilon}$

CASE: $A^{(2)} = 0 \leftarrow$ NOT Biological

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u} \leftarrow \text{constant in time}$$

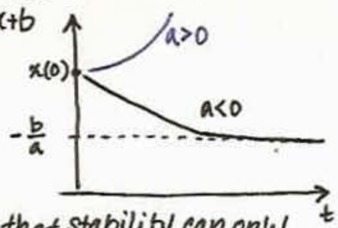
$$A\vec{x}^* + B\vec{u} = 0$$

$$\Rightarrow \vec{x}^* = -A^{-1}B\vec{u}$$

FOR SCALAR CASE:

$$\frac{d}{dt}x = ax + b$$

$$x^* = \frac{-b}{a}$$



observe that stability can only occur if a<0

FOR VECTOR CASE:

$$\frac{d}{dt}\vec{x} = A\vec{x} + B\vec{u}$$

$$\Rightarrow \underbrace{\vec{x} = -A^{-1}B\vec{u}}$$

In order for this to be stable, what must be said about A!

$$\underbrace{\text{Re}\{\text{eigenvalues of } A\} < 0}$$

BUT, WHY EIGENVALUES?

$$A\vec{S}_i = \underset{\uparrow \text{ eigenvalue}}{\lambda}\vec{S}_i \text{ eigenvector}$$

$$A[\vec{S}_1 \vec{S}_2 \cdots \vec{S}_n] = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \lambda_n \end{bmatrix} [S]$$

[redacted]

EIGEN (SPECTRAL) DECOMPOSITION:

[redacted]

$$\vec{x}(t) = \alpha_1(t)\vec{S}_1 + \alpha_2(t)\vec{S}_2 + \ldots + \alpha_n(t)\vec{S}_n$$

$$\Rightarrow \vec{x}(t) = [S]\vec{\alpha}(t)$$

$$\frac{d}{dt}\vec{x} = A\vec{x} + B\vec{u}$$

$$\Rightarrow \frac{d}{dt}\vec{\alpha}(t) = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \lambda_n \end{bmatrix} \vec{\alpha}(t) + [S]^{-1}B\vec{u}$$

$$\Rightarrow \left\{ \begin{array}{l} \frac{d}{dt}\alpha_1 = \lambda_1\alpha_1 + (S^{-1}B\vec{u})_1 \\ \frac{d}{dt}\alpha_2 = \lambda_2\alpha_2 + (S^{-1}B\vec{u})_2 \\ \vdots \\ \frac{d}{dt}\alpha_n = \lambda_n\alpha_n + (S^{-1}B\vec{u})_n \end{array} \right\}$$

$\text{Re}(\lambda_i) < 0 \ \forall i \Rightarrow \alpha_i$'s don't blow-up

CONSIDER A GENERAL, DYNAMIC, NONLINEAR SYSTEM OF EQUATIONS

$$\frac{d}{dt}\vec{x} = F(\vec{x}) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

$$F(\vec{x}) = A^{(1)}\vec{x} + A^{(2)}\vec{x}\otimes\vec{x} + B\vec{u}$$

STEADY-STATE EQN:
Find $\vec{x}^*$ s.t. $F(\vec{x}^*) = 0$

$$F(\vec{x}+\vec{\epsilon}) = F(\vec{x}) + \underset{\uparrow \text{Jacobian of } F(\vec{x})}{J_F(\vec{x})}\vec{\epsilon}$$

(for scalar case: $f(x+\epsilon) \cong f(x) + \frac{\partial f(x)}{\partial x}\epsilon + \text{H.O.T.}$)

SYSTEM ABOUT $\vec{x}^* \Rightarrow$

$$\frac{d}{dt}\vec{x} = F(\vec{x}) \Rightarrow \frac{d}{dt}(\vec{x}^* + \vec{\epsilon}) = F(\vec{x}^* + \vec{\epsilon})$$

$$\Rightarrow \frac{d}{dt}\vec{\epsilon} \cong J_F(\vec{x}^*)\vec{\epsilon}$$

$$J_F(\vec{x})\vec{\epsilon} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

$\vec{x}^*$ is a stable, steady-state IF $\text{Re}\{\text{eigenvalues of } J_F(\vec{x}^*)\} \underline{<0}$

CASE : $A^{(2)} \neq 0$ (Biological Case)

$F(\vec{x}) = A^{(1)}\vec{x} + A^{(2)}\vec{x}\otimes\vec{x} + B\vec{u}$

$J_F(\vec{x}) = A^{(1)} + A^{(2)}(I\otimes\vec{x}) + A^{(2)}(\vec{x}\otimes I)$

STEADY-STATE PROBLEM:

① FIND $\vec{x}^*$ s.t. $A^{(1)}\vec{x}^* + A^{(2)}\vec{x}^*\otimes\vec{x}^* + B\vec{u} = 0$

② VERIFY $\lambda(A^{(1)} + A^{(2)}(I\otimes\vec{x}^* + \vec{x}^*\otimes I))$ have negative real parts

use NEWTON'S METHOD to solve

Problem: Find $\vec{x}^*$ s.t. $F(\vec{x}^*)=0$

FOR THE SCALAR CASE:

$f(x^*)=0$

use Taylor's Expansion for nonlinear f:

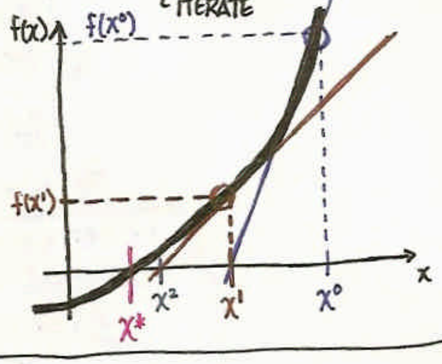
$$\underset{0}{f(x^*)} = f(x) + \frac{df(x)}{dx}(x-x^*) + H.O.T.$$

$$\Rightarrow \frac{df}{dx}(x-x^*) = -f(x)$$

GUESS x^0

COMPUTE x^1: $x^1 - x^0 = -\left(\frac{\partial f}{\partial x}\right)^{-1} f(x^0)$

ITERATE



$\vec{x}^*$ s.t. $F(\vec{x}^*)=0$

Guess at $\vec{x}^0$

$$\left\{\begin{array}{l} J_F(\vec{x}^0)(\vec{x}^1-\vec{x}^0) = -F(\vec{x}^0) \Rightarrow \vec{x}^1 \\ J_F(\vec{x}^1)(\vec{x}^2-\vec{x}^1) = -F(\vec{x}^1) \Rightarrow \vec{x}^2 \\ \quad \vdots \\ \quad \downarrow \end{array}\right\}$$

where

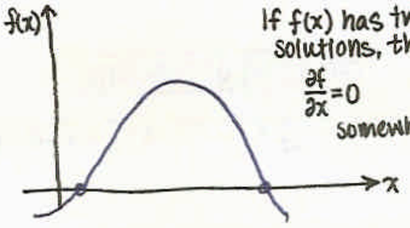
$F(\vec{x}^0) = A^{(1)}\vec{x}^0 + A^{(2)}\vec{x}^0\otimes\vec{x}^0 + B\vec{u}$

$J_F(\vec{x}^0) = A^{(1)} + A^{(2)}(I\otimes\vec{x}^0 + \vec{x}^0\otimes I)$

thus

$$\left\{\begin{array}{l} \left[A^{(1)} + A^{(2)}(I\otimes\vec{x}^0 + \vec{x}^0\otimes I)\right](\vec{x}^1-\vec{x}^0) = -F(\vec{x}^0) \\ \left[A^{(1)} + A^{(2)}(I\otimes\vec{x}^1 + \vec{x}^1\otimes I)\right](\vec{x}^2-\vec{x}^1) = -F(\vec{x}^1) \\ \quad \vdots \\ \quad \downarrow \end{array}\right\}$$

ONCE AGAIN, FOR THE SCALAR CASE:



If $f(x)$ has two solutions, then $\frac{\partial f}{\partial x}=0$ somewhere

SO, FOR $F(\vec{x})=0$ TO HAVE MULTIPLE SOLNS

$\Rightarrow J_F(\vec{x})\vec{v}=0$ For some $\vec{x}$, there is a flat direction

IF $J_F(\vec{x})$ IS NONSINGULAR FOR ALL $\vec{x}$ THEN $F(\vec{x})=0$ HAS ONE SOLUTION

If $A^{(1)} + A^{(2)}(I\otimes\vec{x} + \vec{x}\otimes I)$ is always NONSINGULAR Then STEADY STATE SOLUTION is UNIQUE