Thermal Dynamics and Solar Heating

1. Circuit models
2. Conservation of energy and dynamic models
3. Solar heat gain through windows
4. Solar energy absorbed in walls and roofs
5. Putting windows and walls on an equal computational footing
6. Multi-nodal models
7. Frequency analysis

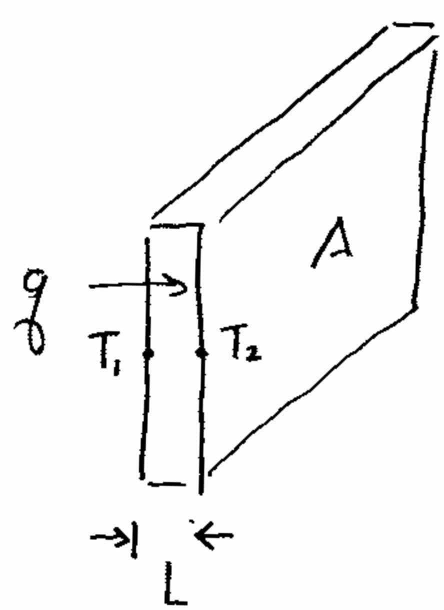
Origins of circuit model

(Mechanical engineers like to use electrical analogies, and vice versa)

We characterize a material, thermally, by its thermal conductivity and heat capacity

Conductivity k $\quad \frac{\text{Btu}}{\text{ft hr °F}}$ or $\frac{\text{W}}{\text{m K}}$

q, heat flow $= \frac{kA}{L}(T_1 - T_2)$ $\quad \frac{\text{Btu}}{\text{hr}}$ or W

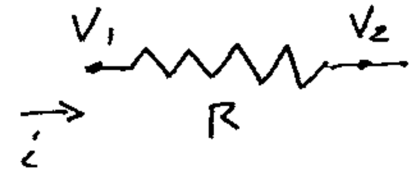


$q = UA(T_1 - T_2)$

where $U = \frac{k}{L}$ $\quad \frac{\text{Btu}}{\text{ft}^2\text{ hr °F}}$ $\quad \frac{\text{W}}{\text{m}^2\text{ K}}$

Electrical analogy

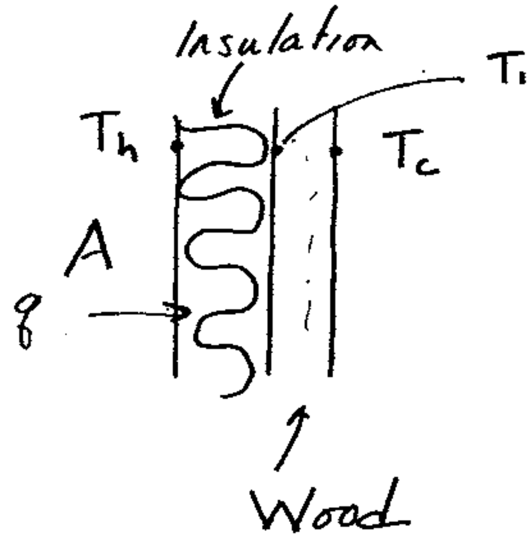
Electricity $\quad i = \dfrac{\Delta V}{R}$ $\quad$ Ohm's law



Heat $\quad q = \dfrac{\Delta T}{R}$ $\quad R = \dfrac{1}{UA} = \dfrac{L}{kA}$

$\dfrac{hr\,{}^\circ F}{Btu}$ or $\dfrac{K}{W}$

Multiple layers (resistances in series)



$$q = \frac{k_{insulation}\,A}{L_{insulation}}\left(T_h - T_1\right)$$

$$= \frac{k_{wood}\,A}{L_{wood}}\left(T_1 - T_c\right)$$

(equality holds in steady state, when no energy is being stored or released from materials)

Re-arrange

$$T_h - T_1 = q \frac{L_{insulation}}{k_{insulation} A} = q R_{insulation}$$

$$T_1 - T_c = q \frac{L_{wood}}{k_{wood} A} = q R_{wood}$$

Add to eliminate T_1

$$T_h - T_c = q \left(R_{insulation} + R_{wood} \right)$$

$$q = \frac{T_h - T_c}{R_{insulation} + R_{wood}}$$

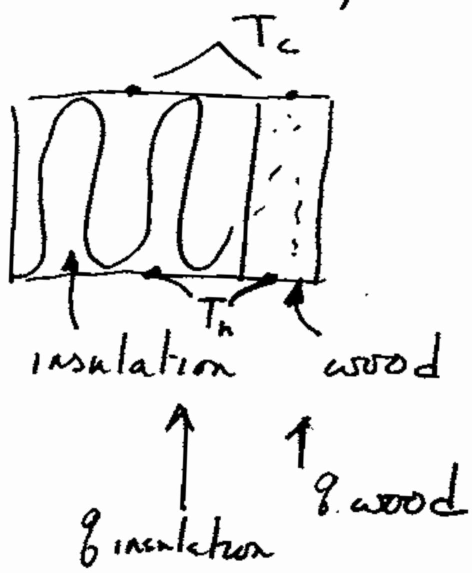
$$= \frac{T_h - T_c}{R_{Total}}$$

$$R_{Total} = R_{insulation} + R_{wood}$$

More layers? $R_{Total} = \sum_i R_i$

Resistances in <u>series</u> add, as any EE will tell you.

Multiple heat-flow paths (resistances in parallel)



$$q = q_{insulation} + q_{wood}$$

$$= \frac{k_{insulation} A_{insulation}}{L_{insulation}} (T_h - T_c)$$

$$+ \frac{k_{wood} A_{wood}}{L_{wood}} (T_h - T_c)$$

$$= \left(\frac{k_{insulation} A_{insulation}}{L_{insulation}} + \frac{k_{wood} A_{wood}}{L_{wood}} \right) (T_h - T_c)$$

$$= \left(\frac{1}{R_{insulation}} + \frac{1}{R_{wood}} \right) (T_h - T_c)$$

$$= \frac{(T_h - T_c)}{R_{Total}}$$

$$\frac{1}{R_{Total}} = \frac{1}{R_{insulation}} + \frac{1}{R_{wood}} = \sum_i \frac{1}{R_i} \text{ in general}$$

Heat capacity

$c \qquad \frac{Btu}{lbm\,^\circ F} \qquad \frac{kJ}{kg\,K}$

We'll use $C = mc$ where m is mass, lbm or kg

C has units $\frac{Btu}{F}$ or $\frac{W}{K}$

Solutions of Conservation-of-Energy Equation for Modeling the Indoor Temperature

Conservation of energy

Change in stored heat $=$ $\sum$ Heat gains and losses

$$C \frac{dT_i}{dt} = \frac{T_o - T_i}{R} + q$$

Units

C	J/K	°C for temperatures on Celsius scale
T_i	°C	
t	sec	K For temperature differences
R	K/W	(could use C here, w/o problem)
q	W	

Steady state

$\frac{dT_i}{dt} = 0$, by definition

$$\frac{T_i - T_o}{R} = q$$

Now allow T_i to change. We'll solve the equation under two special cases, then more generally

Case 1

Let T_o be fixed and let $q = 0$

Assign an initial value to T_i at time $t = 0$

Define $T = T_i - T_o$

Because T_o is fixed, $\frac{dT_o}{dt} = 0$ and $\frac{dT_i}{dt} = \frac{dT}{dt}$

$$RC \frac{dT}{dt} + T = 0 \qquad T\Big|_{t=0} = T_{initial}$$

Define $RC = \tau$ the thermal time constant

$$\tau \frac{dT}{dt} + T = 0$$

$$\frac{dT}{T} = -\frac{1}{\tau}\, dt$$

$\ln T = -\frac{t}{\tau} + A$, where A is a constant of integration

$$T = e^{-t/\tau + A}$$

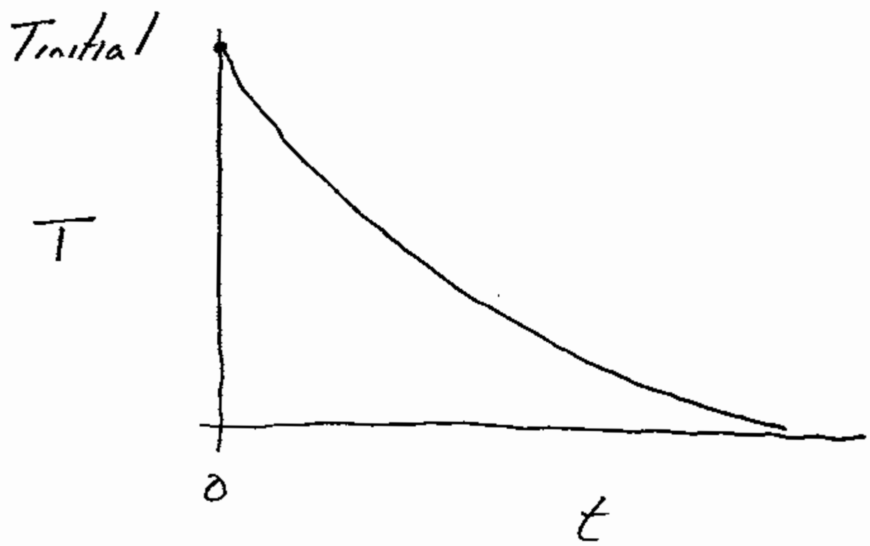
$$= Be^{-t/\tau}$$

$T\big|_{t=0} = B$ but also equals $T_{initial}$ from initial conditions

So, $B = T_{initial}$

$$T = T_{initial}\, e^{-t/\tau}$$

Graphically, the temperature response is an exponential



<u>Case 2</u>

Let T_0 be fixed, but allow q to be a non-zero constant. Let $T_{initial} = (T_i - T_0)\big|_{t=0} = 0$

Solution technique: variation of parameters

$$T = B(t)\, e^{-t/\tau}$$

Notice that B is now a function of time

Recall our equation $\tau \frac{d}{dt}T + T = Rq$

Take a peak at steady state, when $\frac{d}{dt}T = 0$

$T = Rq$, as before, and regardless of $T_{initial}$.

Substitute the assumed solution into the equation

$$\tau\left[e^{-t/\tau}\frac{dB(t)}{dt} + B(t)\left(-\frac{1}{\tau}\right)e^{-t/\tau}\right] + B(t)e^{-t/\tau}$$

$$= Rq.$$

The equation (magically) simplifies

$$\tau e^{-t/\tau}\frac{dB(t)}{dt} = Rq$$

$$\frac{dB(t)}{dt} = \frac{1}{\tau}e^{t/\tau}Rq$$

$$B(t) = \int \frac{1}{\tau}e^{t/\tau}Rq\,dt + D$$

$$= Rq\int e^{t/\tau}d\left(\frac{t}{\tau}\right) + D$$

$$= Rqe^{t/\tau} + D$$

D is another constant of integration

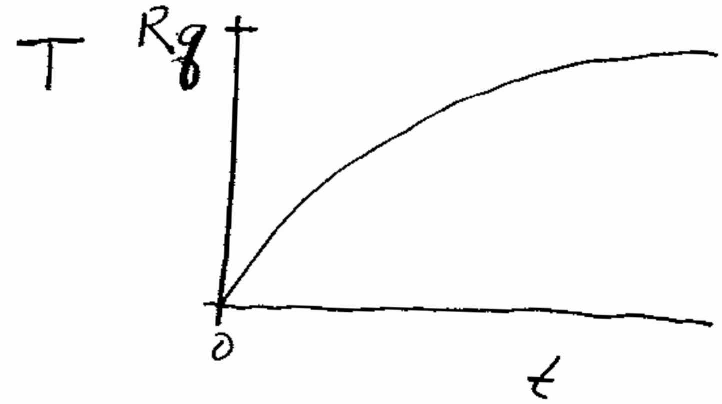
$$T(t) = \left[Rq e^{t/\tau} + D \right] e^{-t/\tau}$$

$T_{\text{initial}} = 0 \Rightarrow D = -Rq$

$$T(t) = Rq \left[e^{t/\tau} - 1 \right] e^{-t/\tau}$$

$$= Rq \left[1 - e^{-t/\tau} \right]$$

Graph it!



This makes good sense visually

Now, let's go on to the general solution, allowing both T_0 and q to vary. Want a practical motivation? The solution will enable us to answer (~~from~~ the perspective of solar energy) the question: "How many windows should I put in my house?"

$$\tau \frac{dT_{in}}{dt} + T_{in} = T_{out} + Rq$$

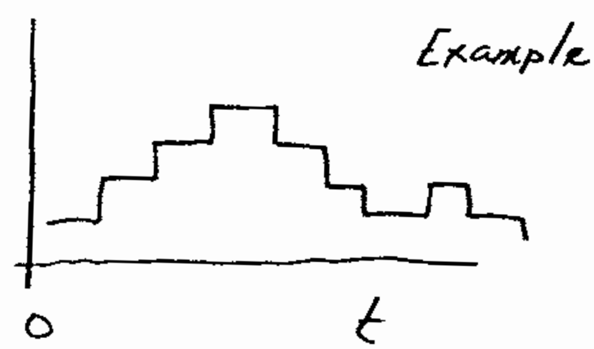
We'll use T_{in} and T_{out} for clarity, because we'll soon add another subscript

First, let T_{out} and q be non-zero but fixed. We can write the solution to the equation on the basis of our previous work

$$T_{in} = \left[(T_{out} + Rq) e^{t/\tau} + D \right] e^{-t/\tau}$$

Now (important! the key to what follows) let both T_{out} and q vary, but only at discrete time intervals (typically hourly)

T_{out} or q



Hour 0.

At this hour we need $T_{out,0}$, q_0, and $T_{in,0}$. Determine the constant D from the initial conditions

$$T_{in,0} = \left[(T_{out,0} + Rq_0) e^{0} + D \right] e^{-0}$$

$$= \left(T_{out,0} + R q_0\right) + D$$

$$D = T_{in,0} - T_{out,0} - R q_0$$

From this instant on. as long as T_{out} and q are fixed at $T_{out,0}$ and q_0,

$$T_{in} = \left[\left(T_{out,0} + R q_0\right) e^{t/\tau} + T_{in,0} - \left(T_{out,0} + R q_0\right)\right] e^{-t/\tau}$$

$$= \left(T_{out,0} + R q_0\right)\left[1 - e^{-t/\tau}\right] + T_{in,0}\, e^{-t/\tau}$$

This is simply a more general form of our earlier solution.

At the end of hour 0 (assuming hourly time steps)

$$T_{in} = \left(\qquad \right)\left[1 - e^{-1/\tau}\right] + T_{in,0}\, e^{-1/\tau}$$

$$= T_{in,1}$$

<u>Hour 1</u>

Repeat the process. In essence, <u>start over</u>, but with new initial conditions $T_{out,1}$, Q_1, $T_{in,1}$

$$T_{in,1} = \left(T_{out,1} + R q_1\right) + D$$

$$D = T_{in,1} - T_{out,1} - R q_1$$

During hour 1,

$$T_{in} = (T_{out,1} + Rq_1)\left[1 - e^{-t/\tau}\right] + T_{in,1}\, e^{-t/\tau}$$

At the end of hour 1

$$T_{in} = (\quad\quad)\left[1 - e^{-1/\tau}\right] + T_{in,1}\, e^{-1/\tau}$$

$$= T_{in,2}$$

There is a pattern here!

At the end of hour n

$$T_{in,n+1} = (T_{out,n} + Rq_n)\left[1 - e^{-1/\tau}\right] + T_{in,n}\, e^{-1/\tau}$$

Solar heat gain through windows

Ref: ASHRAE Handbook of Fundamentals pp 29.27-29.28

Total instantaneous rate of heat gain through glazing equals

Radiation transmitted through glass

+ Inward flow of absorbed solar radiation

+ Heat flow due to indoor-outdoor temperature difference

For single glazing,

$$\frac{q}{A} = I_t \tau + N_{in} \alpha I_t + U_{total}(T_{out} - T_{in})$$

$\frac{W}{m^2}$ $\frac{W}{m^2}$ $\frac{W}{m^2 K}$ K

(I'll try to use $\frac{q}{A}$ for W/m^2, q for W)

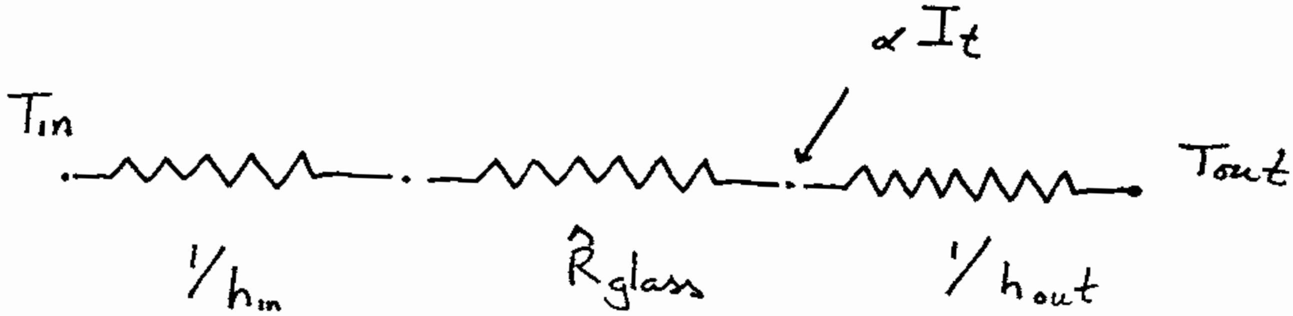
I_t is total solar radiation, direct and diffuse (from sun and sky)

τ - transmissivity or transmittance

α - absorptivity

U_{total} - U value for glazing and air layers on each side (and between multiple panes)

N_{in} is the inward-flowing fraction of absorbed radiation



$\hat{R} = \dfrac{1}{U}$ no area

h = coefficient of heat transfer by long-wave radiation and convection at the (inner or outer) surface W/m^2K

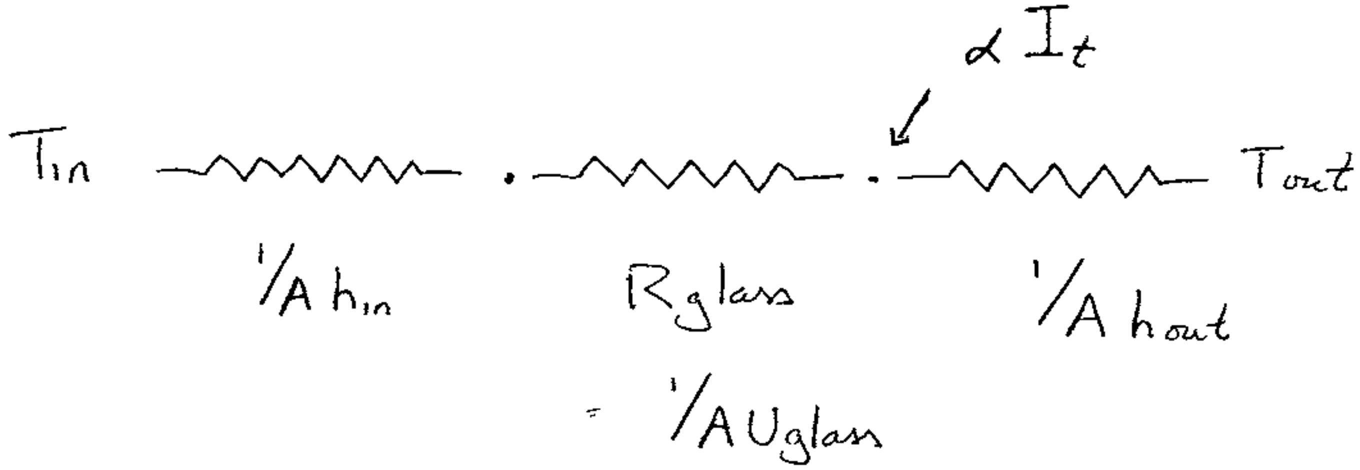
$$\frac{1}{U_{window}} = \hat{R}_{window} = \frac{1}{h_{in}} + \hat{R}_{glass} + \frac{1}{h_{out}}$$

The fraction N_{in} can be determined by inspection or by the concept of a voltage divider

$$N_{in} = \frac{1/h_o}{\hat{R}_{window}} = \frac{U_{window}}{h_{out}}$$

(Make an argument in terms of fractions flowing in and out, summing to 1.0)

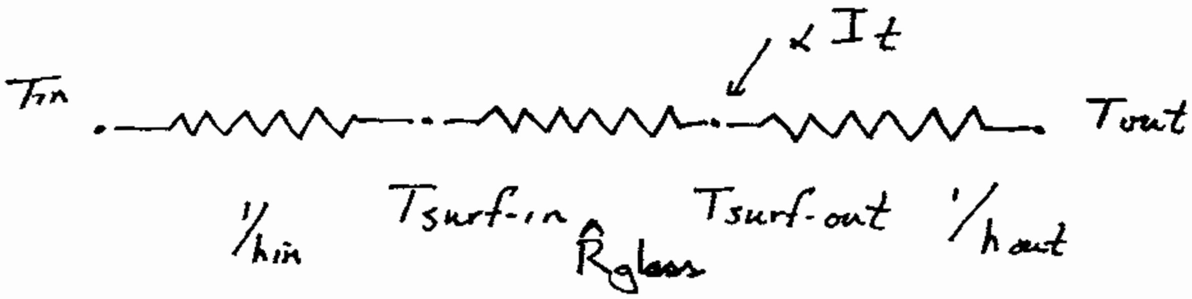
$$\frac{q}{A} = \left(\hat{\tau}_{window} + \alpha_{window}\frac{U_{window}}{h_{out}}\right)I_t + U_{window} \times (T_{out} - T_{in})$$



$$R_{window} = \frac{1}{A h_{in}} + R_{glass} + \frac{1}{A h_{out}} = \frac{1}{A U_{window}}$$

$$N_{in} = \frac{1/A h_{out}}{R_{window}} \quad \frac{U_{window}}{h_{out}}$$

Voltage divider, worked out



No solar

$$\frac{q}{A} = \frac{T_{in} - T_{out}}{\hat{R}_{window}}$$

$$= \frac{T_{in} - T_{surf\text{-}in}}{1/h_{in}} = \frac{T_{surf\text{-}in} - T_{surf\text{-}out}}{\hat{R}_{glass}} = \frac{T_{surf\text{-}out} - T_{out}}{1/h_{out}}$$

$\therefore$

$$T_{in} - T_{surf\text{-}in} = \frac{1/h_{in}}{\hat{R}_{window}} (T_{in} - T_{out}) = \frac{U_{window}}{h_{in}} (T_{in} - T_{out})$$

$$T_{surf\text{-}in} - T_{surf\text{-}out} = \frac{\hat{R}_{glass}}{\hat{R}_{window}} (T_{in} - T_{out})$$

$$T_{surf\text{-}out} - T_{out} = \frac{1/h_{out}}{\hat{R}_{window}} (T_{in} - T_{out}) = \frac{U_{window}}{h_{out}} (T_{in} - T_{out})$$

Let's look at some values.

2001 ASH HoF Table 1, p 25.2

surface conductances and resistances for still and moving air

1. Convection and radiation in parallel. conductances <u>add</u>

   $h_r \approx 5\ W/m^2 K$

   h_c 1.2 - 4.3 as f (orientation)

2. Total for still air

   h 6.1 - 9.2

   R 0.11 - 0.16 $\frac{m^2 K}{W}$

3. For moving air

   h 22.7 - 34

   R 0.03 - 0.044

4 What about glass itself?

   $k_{glass} \approx 1\ W/m\,K$

   For 3 mm (1/8")

   $U = \frac{1}{.003} = 333$

   $R = 0.003\ m^2 K/W$

T_{in} —/\/\/\/\— • —/\/\/\/\— • —/\/\/\/\— T_{out}

0.11 - 0.16   0.003   0.03 - 0.044

Sum is ~ 0.12 + .003 + .04 = 0.16(3)

0.16 × 5.67 = R 0.9 in IP units

Glass is negligible

Even w/ boundary layers, a single layer is ~ 1/5 of 2.5 cm of EPS

Define the solar heat gain coefficient as

$$SHGC = \tau + \alpha N_{in}$$

This applies to any glazing.

Then

$$\frac{q}{A} = SHGC\, I_t + U_{window}\,(T_{out} - T_{in})$$

For single pane, $SHGC = \tau + \alpha \dfrac{U_{window}}{h_{out}}$

For others, it can be modeled or measured. See ASHRAE handout.

Alternatively, define

$$SHGF = \left(\tau + \alpha \frac{U_{window}}{h_{out}}\right) I_t$$

the solar energy as measured on the inside of a single pane of standard 3 mm clear glass.

For other glazings,

$$\frac{q}{A} = SC \times SHGF + U_{window}\,(T_{out} - T_{in})$$

What have we forgotten? Solar energy absorbed by walls and the roof. Light-colored surface? Dark? It makes a difference!

Sol-Air Temperature

What's the idea? Replace Tout with a new temperature that accounts for solar energy absorbed by opaque surface

Ref: ASHRAE HoF 28.5-28.6

" Sol-air temperature is the temperature of the outdoor air that, in the absence of all radiation changes, gives the same rate of heat entry into the surface as would the combination of incident solar radiation, radiant energy exchange with the sky and other outdoor surroundings, and convective heat exchange with the outdoor air."

The heat balance at a sun lit surface gives the heat flux into the surface as

$$\frac{q}{A} = \alpha I_t + h_{out}(T_{out} - T_s) - \epsilon \Delta R$$

where the new terms are

ϵ (hemispherical) emittance of the surface

T_s outer surface temperature

ΔR . difference between long-wave radiation incident on the surface from the sky and surroundings and radiation emitted by a blackbody at the outdoor air temperature, W/m^2

We define the sol-air temperature by assuming the rate of heat transfer can be expressed in terms of $T_{sol\text{-}air}$

$$\frac{q}{A} = h_{out} \left(T_{sol\text{-}air} - T_{surface} \right)$$

where

$$T_{sol\text{-}air} = T_{out} + \frac{\alpha}{h_o} I_t - \frac{\epsilon \Delta R}{h_o}$$

Horizontal surfaces

For horizontal surfaces that receive long-wave radiation from the sky only, $\Delta R \doteq 63\, W/m^2$. If $\epsilon = 1$ and $h_{out} \doteq 17.0 \frac{W}{m^2 K}$, the long-wave correction term is $\sim -3.9\,°C$.

Vertical surfaces

· Because vertical surfaces receive long-wave radiation from the ground and surrounding buildings as well as from the sky, accurate ΔR values are difficult to determine. When solar radiation intensity is high, surfaces of terrestrial objects usually have a higher temperature than the outdoor air; thus, their long-wave radiation compensates to some extent for the sky's low emittance. Therefore, it is common practice to assume $\Delta R = 0$ for vertical surfaces.

Tabulated values

Assume $\dfrac{\epsilon \Delta R}{h_{out}} = -3.9\ K$ horizontal

$= 0\ K$ vertical

Surface colors

$\dfrac{\alpha}{h_{out}} = 0.026$ light-colored surfaces

$= 0.052$ dark-colored surfaces

h_{out} has units $\dfrac{W}{m^2 K}$

$\dfrac{\alpha}{h_o}$ $\dfrac{m^2 K}{W}$

For $h_{out} = 17\ W/m^2 K$

$\dfrac{\alpha}{h_{out}} = 0.026 \Rightarrow \alpha = 0.44$

$= 0.052 \qquad = 0.88$

Note: $17 \dfrac{W}{m^2 K} \approx 3 \dfrac{Btu}{hr\ ft^2\ °F}$ or

a thermal resistance of $\frac{1}{3} \dfrac{hr\ ft^2\ °F}{Btu}$ "R 1/3"

Try October, for example, and compute 24-hour average

Compute daily total SHGF and multiply by $\frac{1}{0.87} = 1.15$ to convert SHGF to I_t

For directions other than N, S, or horizontal, daily total requires two complementary half-day totals

Ex: E or W $\left(1964\ \frac{Wh}{m^2} + 279\ \frac{Wh}{m^2}\right) \times 1.15 = 2579\ \frac{Wh}{m^2}$

$$T_{sol\text{-}air} = T_{out} + \frac{\alpha}{h_{out}} \frac{I_{daily\ total}}{24\ h} - \frac{\epsilon\, \Delta R}{h_{out}}$$

E or W

$$T_{sol\text{-}air} - T_{out} = \frac{\alpha}{h_{out}} \frac{2579\ Wh/m^2}{24\ h}$$

$= 2.8$ K light-colored surface

$= 5.6$ K dark-colored surface

Important!

Horizontal

$$T_{sol\text{-}air} - T_{out} = \frac{\alpha}{h_{out}} \left(\frac{1704 \times 2 \times 1.15}{24}\right) - 3.9\ K$$

$= 0.3$ K light-colored surface

$= 4.6$ K dark-colored surface

North

$T_{sol\text{-}air} - T_{out}$: 0.7 K light-colored surfaces
1.4 K dark-colored surfaces

South

$T_{sol\text{-}air} - T_{out}$: 6.2 K light-colored surfaces
12.4 K dark-colored surfaces

2/26/03

Putting walls and windows on equal footing.

1. First, we will make walls like windows, to permit a direct comparison of single surfaces

2 Second, we will make windows like walls, to allow us to deal with realistic buildings

Advanced topics

The concept of $T_{sol\text{-}air}$ helps us think about energy and building façades in an integrated manner. It quantifies the difference between light- and dark-colored surfaces. Because it puts absorbed solar energy on a temperature scale, it appeals to our intuition. But it is not quite enough. Why?

1. We cannot use it, yet, in our solution to the equation that governs the time history of T_{in}.

2. We cannot directly compare walls with windows, or evaluate glazing-wall systems (Trombe walls) or decide how much wall insulation is optimal.

Here's the problem:

$$\frac{\dot{q}}{A} = h_{out}\left(T_{sol\text{-}air} - T_s\right)$$

This equation includes the outside surface temperature, T_s. It is more useful to express heat flux in terms of T_{out} and T_{in}, as is done in the absence of radiant energy absorbed at the surface.

We can eliminate T_s in two ways:

1. comparison of a wall with a window
2. brute force

1. Let's take the easy route first

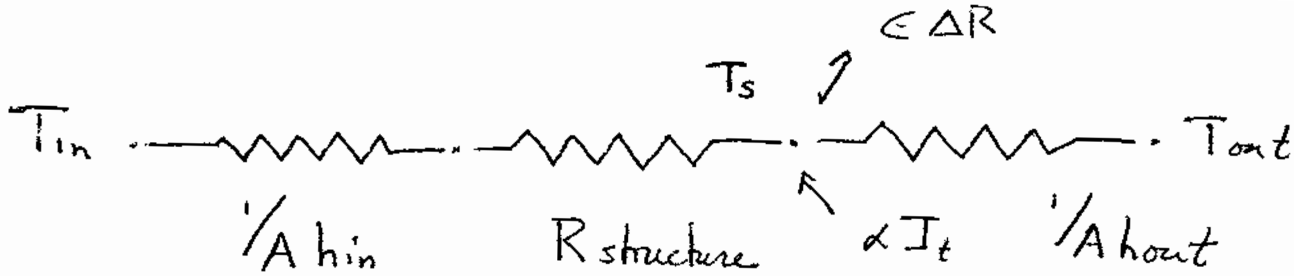
Window

$$\frac{q}{A} = SC\left(\tau + \alpha_{window}\frac{U_{window}}{h_{out}}\right)I_t + U_{window}(T_{out} - T_{in})$$

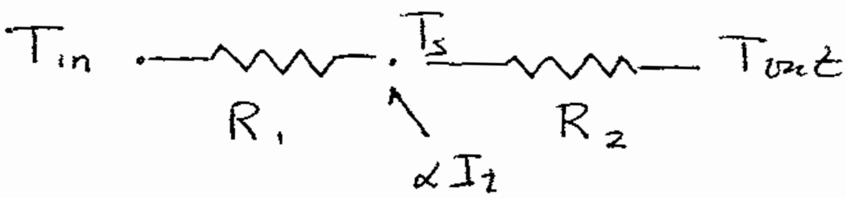
Wall, by analogy

$$\frac{q}{A} = \alpha_{wall}\frac{U_{wall}}{h_{out}}I_t + U_{wall}(T_{out} - T_{in})$$

2. Brute force - only if skeptics rule the day



Neglect $\epsilon \Delta R$



$$R_1 + R_2 = R_{wall} = \frac{1}{U_{wall} A}$$

At night

$$q = \frac{T_{out} - T_s}{R_2} = \frac{T_{out} - T_{in}}{R_1 + R_2}$$

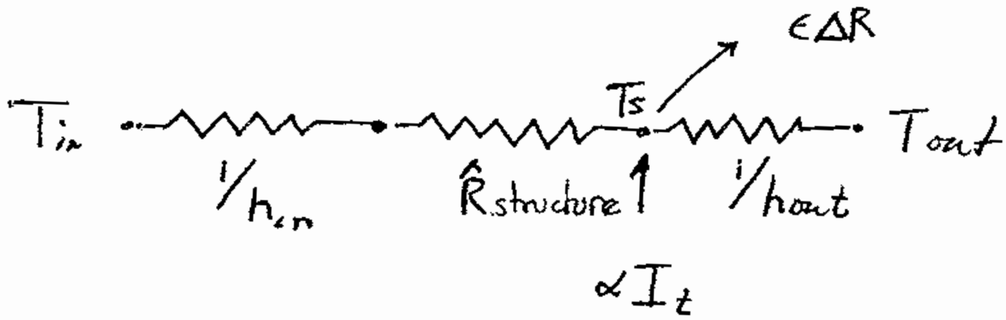
$$(T_{out} - T_s) = \frac{R_2}{R_1 + R_2} (T_{out} - T_{in}) = \frac{U_{wall}}{h_{out}} (T_{out} - T_{in})$$

Recall our previous definition

$$\frac{q}{A} = \alpha I_t + h_{out} (T_{out} - T_s) - \epsilon \Delta R$$

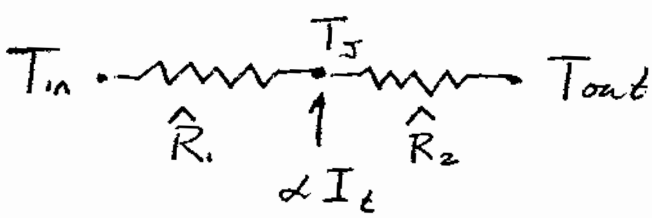
and note we can substitute and obtain $U_{wall} (T_{out} - T_{in})$

Let's return to our circuit model, applying it to a wall



Let's neglect $\epsilon \Delta R$ for simplicity, and because it is about zero for vertical surfaces.

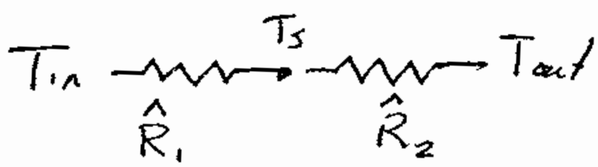
Simplify the circuit



$$\hat{R}_1 = \frac{1}{h_{in}} + \hat{R}_{structure} \qquad \hat{R}_1 + \hat{R}_2 = \hat{R}_T = \frac{1}{U_{wall}}$$

$$\hat{R}_2 = \frac{1}{h_{out}}$$

First, what do we expect at night, when there is no solar gain? This will serve as a check on our final answer.



$$(T_{out} - T_s) = \frac{\hat{R}_2}{\hat{R}_1 + \hat{R}_2} (T_{out} - T_{in})$$

This can be seen easily from

$$\frac{q}{A} = \frac{T_{out} - T_s}{\hat{R}_2} = \frac{T_{out} - T_{in}}{\hat{R}_1 + \hat{R}_2}$$

Now include αI_t. The sum of the heat flows at the outer surface (taking flow into the surface as positive) must equal zero. The math is here for the record, but can be taken on faith if desired.

$$\alpha I_t + \frac{1}{\hat{R}_2}(T_{out} - T_s) + \frac{1}{\hat{R}_1}(T_{in} - T_s) = 0$$

$$\alpha I_t + \frac{1}{\hat{R}_2}(T_{out} - T_s) + \frac{1}{\hat{R}_1}\left[(T_{in} - T_{out}) - (T_s - T_{out})\right] = 0$$

$$\alpha I_t + \left(\frac{1}{\hat{R}_1} + \frac{1}{\hat{R}_2}\right)(T_{out} - T_s) = \frac{1}{\hat{R}_1}(T_{out} - T_{in})$$

$$\alpha I_t + \frac{\hat{R}_1 + \hat{R}_2}{\hat{R}_1 \hat{R}_2}(T_{out} - T_s) = \frac{1}{\hat{R}_1}(T_{out} - T_{in})$$

$$(T_{out} - T_s) = \frac{\hat{R}_2}{\hat{R}_1 + \hat{R}_2}(T_{out} - T_{in}) - \alpha \frac{\hat{R}_1 \hat{R}_2}{\hat{R}_1 + \hat{R}_2} I_t$$

In the absence of solar radiation,

$$(T_{out} - T_s) = \frac{\hat{R}_2}{\hat{R}_1 + \hat{R}_2}(T_{out} - T_{in})$$

as expected!

Now substitute into

$$\frac{q}{A} = \alpha I_t + h_{out}(T_{out} - T_s)$$

Recognize that by definition $\frac{1}{h_{out}} = \hat{R}_2$

$$\frac{q}{A} = \alpha I_t + \frac{1}{\hat{R}_2}\left(T_{out} - T_s\right)$$

$$= \alpha I_t + \frac{1}{\hat{R}_2}\left[\frac{\hat{R}_2}{\hat{R}_1+\hat{R}_2}\left(T_{out} - T_{in}\right) - \alpha \frac{\hat{R}_1\hat{R}_2}{\hat{R}_1+\hat{R}_2} I_t\right]$$

$$= \alpha\left(1 - \frac{\hat{R}_1}{\hat{R}_1+\hat{R}_2}\right) I_t + \frac{1}{\hat{R}_1+\hat{R}_2}\left(T_{out} - T_{in}\right)$$

$$= \alpha \frac{\hat{R}_2}{\hat{R}_1+\hat{R}_2} I_t + \frac{1}{\hat{R}_1+\hat{R}_2}\left(T_{out} - T_{in}\right)$$

$$= \alpha \frac{U_{wall}}{h_{out}} I_t + U_{wall}\left(T_{out} - T_{in}\right)$$

as before!

Examples

1. First, a wall. The question: How much insulation to put in a south-facing wall? A north-facing wall? Does the sun make a difference? As part of this class, we might think of this issue as an integration of energy, structure and façade. In developing countries, where insulating material may be scarce and expensive, it is a practical and important matter.

Let $U_{wall} = 0.5 \frac{W}{m^2 K}$ (~ 2" rigid foam, with boundary layers ignored at ±10% accuracy)

$h_{out} = 17 \frac{W}{m^2 K}$

Let $\alpha = 0.44$ (light-colored surface)

$T_{out} - T_{in} = 10\ K$

$I_t = 500 \frac{W}{m^2}$

Then $\frac{q}{A}\Big|_{wall} = \underbrace{6.5 \frac{W}{m^2}}_{\text{solar gain}} - \underbrace{5.0 \frac{W}{m^2}}_{\text{conductive loss}} = +1.5 \frac{W}{m^2}$

With the sun shining on it, our wall gains energy, even though it is colder outside than in.

Now let's have some fun and reduce $\hat{R}_1$ from 2 to 1 $\frac{m^2 K}{W}$ - i.e., remove some insulation.

What do you expect? More heat loss (given the temperature difference) - or not ??

2"

$U = \frac{.029}{.0508} = 0.57$ $R = 1.7517$

boundary layers add $\doteq 0.16$

$R_{Tot} \sim 1.91$

$U \doteq 0.52$

1" $U = \frac{.029}{.0254} = 1.1417$ $R = 0.8759$

add $\doteq 0.16$

$R_{Tot} = 1.04$ $U = 0.9654 \doteq 0.97$

$$\left.\frac{q}{A}\right|_{wall} = 13\ \frac{W}{m^2} - 10\ \frac{W}{m^2} = +3\ \frac{W}{m^2}$$

Under these conditions, <u>less</u> insulation is better (!)

For a dark-colored surface, we have

$$\frac{q}{A} = 13\ \frac{W}{m^2} - 5\ \frac{W}{m^2} = 8\ W/m^2 \qquad U_{wall} = 0.5\ \frac{W}{m^2 K}$$

$$= 26\ \frac{W}{m^2} - 10\ \frac{W}{m^2} = 16\ W/m^2 \qquad = 1.0\ \frac{W}{m^2 K}$$

The heat balance is even more positive if $\alpha = 0.88$, appropriate for a dark-colored surface. <u>But</u> we must also account for nights and cloudy days, when the sun is not shining on the wall. What about a 24-hour heat balance? Or a heat balance over an entire winter? Over a Beijing winter, for example, more insulation is better.

Two thoughts:

1. For more than snapshots, this problem begs for a computer program.

2. It would be worthwhile to actively vary the level of insulation in a wall, via movable panels or some as-yet uninvented high-tech solution, like micro-shutters. The insulation level could be reduced during sunny hours.

Practical issues (an aside)

Mounting insulation on walls in China and Pakistan, where wood is very expensive

Double-layer block walls

2. Let's compare a wall with a window

Wall: $\alpha_{wall} = 0.88$

$U_{wall} = 0.5 \ W/m^2K$

$h_{out} = 17 \ W/m^2K$

Window: single pane, standard glass

$SC = 1$

$$\left(\tau + \alpha_{window} \frac{U_{window}}{h_{out}}\right) = 0.86$$

$U_{window} = 5 \ W/m^2K$

$$\frac{q}{A}\bigg|_{wall} = 0.026\, I_t + 0.5\,(T_{out} - T_{in})$$

$$\frac{q}{A}\bigg|_{window} = 0.86\, I_t + 5\,(T_{out} - T_{in})$$

If $T_{out} \approx T_{in}$, the glass is better for heat gain

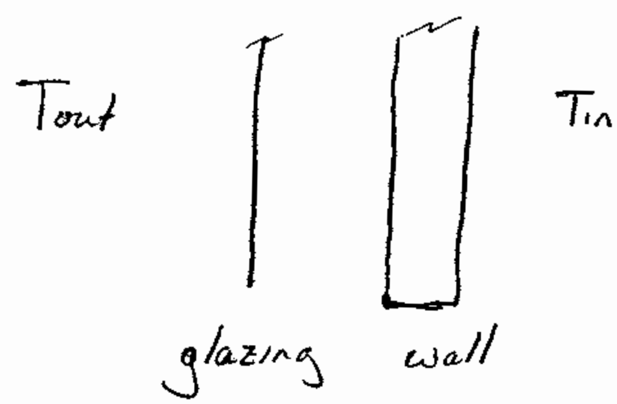
$I_t \approx 0$, the wall is better

How about a better window? Typically, reducing the conductive heat transfer also reduces the shading coefficient. For example, consider an SC of 0.5 and $U_{window} = 2 \frac{W}{m^2 K}$

Summary

$T_{out} - T_{in}$, K	I_t	$\frac{q}{A}\Big\vert_{wall}$	$\frac{q}{A}\Big\vert_{window}$	$\frac{q}{A}\Big\vert_{better\ window}$
	W/m²	W/m²	W/m²	W/m²
−10	500	10	385	197.5
−10	20	−4.4	−32.6	−11.3

3. Let's try a Trombe wall, a combination of glazing and an opaque wall. We'll neglect longwave radiation



We could work out an equivalent circuit for this system, but there is a simpler, yet still precise, approach. We will model this as a wall, but with two modifications:

1. Reduction of incident solar heat gain on the wall, to account for the outer glazing

2. Reduction of the outer air-layer conductance, h_{out} for the same reason.

To reduce the solar radiation, multiply α_{wall} by

$$SC \cdot SHGF$$

i.e.

plain wall	Trombe wall
$\alpha_{wall} I_t$	$\alpha_{wall} \cdot SC \cdot SHGF$ or $\alpha_{wall} \cdot SHGC \cdot I_t$

Denote a reduction in h_{out} by a new heat-transfer coefficient, h^*_{out}. In our terminology,

Plain wall	Trombe wall
h_{out}	h^*_{out}
U_{wall}	U^*_{wall}

Example. Make a Trombe wall with a single layer of glazing

α_{wall} : 0.88

SHGC : 0.86

h_{out} : 17 $W/m^2 K$

$1/h_{out}$: 0.06 $m^2 K/W$

h^*_{out} : $U_{\substack{\text{single-pane} \\ \text{standard window}}} \cong 6 \dfrac{W}{m^2 K}$

$1/h^*_{out}$ = 0.16 $m^2 K/W$ If 0.12 + 0.12 + 0.04, then 0.28 over est due to radn

U_{wall} = 0.5 $W/m^2 K$

U^*_{wall} = 0.48 $W/m^2 K$

$$\left.\frac{q}{A}\right|_{\text{Trombe wall}} = 0.058\, I_t + 0.48\,(T_o - T_i)$$

$$\left.\frac{q}{A}\right|_{\text{wall}} = 0.026\, I_t + 0.50\,(T_o - T_i)$$

The Trombe wall takes better advantage of the sun and reduces conductive losses.

Trombe Wall Calculator

Alphawall	0.88	
SHGC	0.86	
hout	17	W/m2K
hout*	4.00	
Uwall	1	W/m2K
Uwall*	0.84	
Uwall2	0.5	W/m2K
Uwall2*	0.46	
Uwindow	6	
Uwindow2	3	
delT	-10	K
It1	500	W/m2
It2	20	

	q/A		W/m2			
	window	window	wall	wall	Trombe	Trombe
	U=6.0	U=3.0	U=1.0	U=0.5	U=1.0	U=0.5
500 W/m2	370.0	339.8	15.9	7.9	84.0	45.6
20 W/m2	-42.8	-15.2	-9.0	-4.5	-4.7	-2.6

What's best?

Need annual climate data and cost of materials and energy

Trombe Wall Calculator

Alphawall	0.88	
SHGC	0.86	
hout	17	W/m2K
hout*	4.43	
Uwall	1	W/m2K
Uwall*	0.86	
Uwall2	0.5	W/m2K
Uwall2*	0.46	
Uwindow	6	
Uwindow2	3	
delT	-10	K
It1	500	W/m2
It2	20	

?

	q/A window U=6.0	window U=3.0	W/m2 wall U=1.0	wall U=0.5	Trombe U=1.0	Trombe U=0.5
500 W/m2	370.0	339.8	15.9	7.9	76.5	41.2
20 W/m2	-42.8	-15.2	-9.0	-4.5	-5.2	-2.8

What's best?
Need annual climate data and cost of materials and energy

$1/h_{out} \approx 0.06$

h_c r h_r don't overdue it.

$1/h^*_{out} \sim 1/6^+$

Correcting our prediction of T_{in} to account for solar energy absorbed by opaque walls

1. What we now have:

$$T_{in,n+1} = \left(T_{out,n} + R q_n\right)\left(1 - e^{-\Delta t/\tau}\right) + T_{in,n}\, e^{-\Delta t/\tau}$$

where Δt, the time step, has been taken as 1 hour.

This accounts, properly, for solar energy transmitted through and absorbed by the window.

2. To account for solar heat absorbed by walls, we have two choices:

   a. Provide a heat gain term similar to q_n for each wall

   b. Replace $T_{out,n} + R q_n$ with a sum of sol-air temperatures, including one for the window.

If we take the first approach we have

$$T_{in,n+1} = \left[T_{out,n} + R q_n + R \sum_{i}^{walls}\left(\frac{\alpha_i}{h_{out,i}} I_i - \frac{\varepsilon_i}{h_{out,i}} I_{LW,i}\right)\right]\left(1 - e^{-1/\tau}\right)$$
$$+ T_{in,n}\, e^{-1/\tau}$$

Note that $\varepsilon I_{LW} \approx 65 \frac{W}{m^2}$ for horizontal surfaces and ≈ 0 for vertical surfaces

If we take the second approach, we have a simpler expression.

$$T_{in,n+1} = R\left[\sum_{i}^{walls} \frac{T_{SA\,wall,i}}{R_{wall,i}} + \frac{T_{SA\,window}}{R_{window}}\right]\left(1 - e^{-\Delta t/\tau}\right)$$

$$+ \; T_{in,n}\, e^{-\Delta t/\tau}$$

where $$T_{SA\,wall} = T_{out} + \frac{\alpha_{wall} I_t}{h_{out}} - \underbrace{\frac{\epsilon_{wall} I_{LW}}{h_{out}}}_{\text{if horizontal}}$$

$$T_{SA\,window} = T_{out} + SC \cdot \left(\frac{\alpha_{window}}{h_{out}} + \frac{\tau}{U_{window}}\right) I_t - \underbrace{\frac{\epsilon_{window} I_{LW}}{h_{out}}}_{\text{if horizontal}}$$

The second approach is simply a weighted average of the sol-air temperature for each surface

or $$T_{out} + \frac{SHGC}{U_{window}} I_t - \underbrace{\frac{\epsilon_{window} I_{LW}}{h_{out}}}_{\text{if horizontal}}$$

12/4/01

Adding airflow to TSA calculations

$$T_{in,n+1} = R\left[\frac{T_{out}}{R_{airflow}} + \sum_{i}^{walls}\frac{TSA_{wall,i}}{R_{wall,i}} + \frac{TSA_{window}}{R_{window}}\right]$$

For TSA, think of absorbed inward flowing fraction of I, divided by U, which is equivalent to multiplying by R

$$\Delta T = qR$$

3. The explanation

a. The sol-air temperature increases T_{out} to account for heat gains. We've seen this already for walls.

b. It makes sense to take an R-weighted average of T_{SA} for each surface.

c. Consider T_{SA} for the window in detail

First

Recall the window heat balance, without long-wave radiation

$$\frac{q}{A}\Big|_{window} = SC \cdot SHGC \cdot I_t + U_{window}(T_{out} - T_{in})$$

$$= SC \cdot \left(\tau + \frac{\alpha_{window} U_{window}}{h_{out}}\right) I_t$$

$$+ U_{window}(T_{out} - T_{in})$$

$$= U_{window}\left[T_{out} + SC\left(\frac{\tau}{U_{window}} + \frac{\alpha_{window}}{h_{out}}\right) I_t - T_{in}\right]$$

$$= U_{window}(T_{SA\,window} - T_{in})$$, again without I_{LW}

Second

Compare with what we've already derived as the impact of solar energy through glass:

$$T_{out} + Rq = T_{out} + R\left[SC\left(\tau + \frac{\alpha_{window}}{h_{out}} U_{window}\right)\right] I_t \cdot A_{window}$$

$$= T_{out} + R\left[SC\left(\frac{\tau}{U_{window}} + \frac{\alpha_{window}}{h_{out}}\right)\right] I_t \, U_{window} A_{window}$$

$$U_{window} A_{window} = \frac{1}{R_{window}}$$

$$T_{out} + Rq = T_{out} + SC\left(\frac{\tau}{U_{window}} + \frac{\alpha_{window}}{h_{out}}\right) I_t \frac{R}{R_{window}}$$

$$= T_{out}\left(1 - \frac{R}{R_{window}}\right) + \left[T_{out} + SC\left(\frac{\tau}{U_{window}} + \frac{\alpha_{window}}{h_{out}}\right) I_t\right] \frac{R}{R_{window}}$$

$$= T_{out}\left(1 - \frac{R}{R_{window}}\right) + T_{SA\,window} \frac{R}{R_{window}}$$

This is exactly what we can derive from our new, most general expression if there is negligible solar gain on the walls. In that case,

$$T_{SA\,wall} \approx T_{out}$$

and

$$T_{out}\left(1 - \frac{R}{R_{window}}\right) + T_{SA\,window}\frac{R}{R_{window}}$$

$$= T_{out}\sum_{i=1}^{walls}\frac{R}{R_{wall,i}} + T_{SA\,window}\frac{R}{R_{window}}$$

where $$1 = R\left[\sum_{i=1}^{walls}\frac{1}{R_{wall,i}} + \sum\frac{1}{R_{window}}\right]$$

$$R\left[\sum_{i}^{walls}\frac{T_{SA\,wall,i}}{R_{wall,i}}+\frac{T_{SA\,window}}{R_{window}}\right]$$

$$=R\left[\sum_{i}^{walls}\frac{T_{out}}{R_{wall\,i}}+\frac{T_{SA\,window}}{R_{window}}\right]$$

$$=T_{out}\,R\sum_{i}^{walls}\frac{1}{R_{wall\,i}}+T_{SA\,window}\frac{R}{R_{window}}$$

$$=T_{out}\,R\left(\frac{1}{R}-\frac{1}{R_{window}}\right)+T_{SA\,window}\frac{R}{R_{window}}$$

where $\frac{1}{R}=\frac{1}{R_{window}}+\sum_{i}^{walls}\frac{1}{R_{wall,i}}$

$$=T_{out}\left(1-\frac{R}{R_{window}}\right)+T_{SA\,window}\frac{R}{R_{window}}$$

Heat storage calcs

Water 4.2 kJ/kg C

1000 kg/m3

Air 1.0 kJ/kg C

1.2 kg/m3

Say 10 kg water (10 l)
10 l air

Then 42 kJ/C for water

0.012 kJ/C air

Negligible if equal volumes, because 10^3 on density

The real issue here is whether the mass is inside or outside the insulation, and whether the sun hits the mass or something lightweight.

Multi-node models

Impact of insulation - in or out

Impact of location of mass - what if direct radiation does not hit the mass?

A long answer to a short question:

"What about the heat capacity of the air in my elf house? Do I add it to that of my thermal mass? Average it? Forget it? If I calculate $m \cdot c = C$

$$\underset{kg}{m} \cdot \underset{\frac{kJ}{kg\,K}}{c} = \underset{\frac{kJ}{K}}{C}$$

for my mass and for the air,

$C_{mass} \doteq 6 \times C_{air}$."

The answer. In fact, two answers.

1. The model (just a representation of reality, and a simple one at that) has only two temperatures: T_{out} and T_{in}. T_{in} is associated with the dominant thermal mass. The model assumes all of the mass is at the same temperature. Because $C_{mass} \gg C_{air}$, forget about C_{air}.

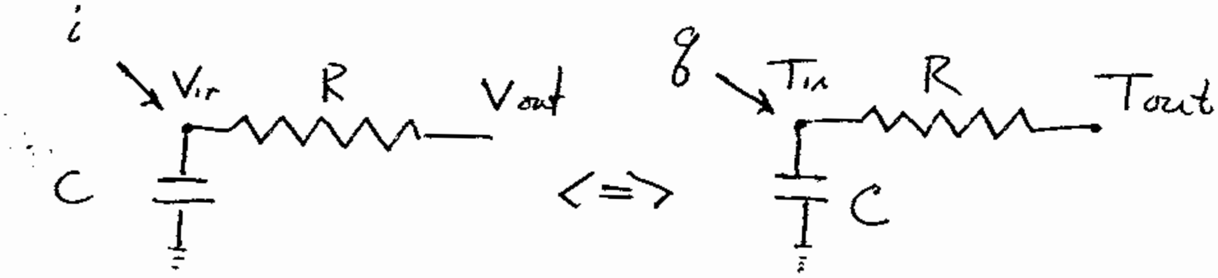
But what if we realize that a model with just one C is not good enough? That $T_{air} \neq T_{mass}$, inside a building, at least at some times. On to answer #2

Complex models

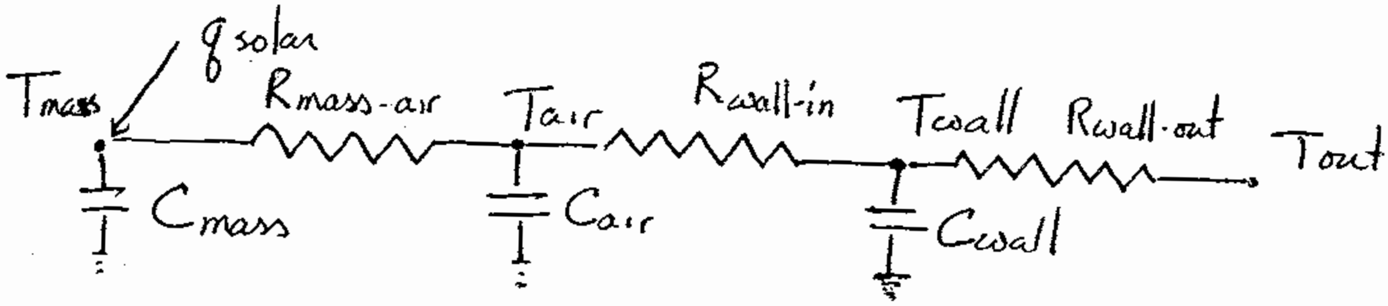
1. Recognize that our simple equation

$$C \frac{dT_{in}}{dt} = \frac{T_{out} - T_{in}}{R} + q$$

has a circuit analog



2. We can model our elf house in any way we think includes the right amount of detail. For example



3 We use conservation of energy to write down one differential equation for each thermal capacitance

$$C_{mass} \frac{dT_{mass}}{dt} = \frac{T_{air} - T_{mass}}{R_{mass\text{-}air}} + q_{solar}$$

$$C_{air} \frac{dT_{air}}{dt} = \frac{T_{mass} - T_{air}}{R_{mass\text{-}air}} + \frac{T_{wall} - T_{air}}{R_{wall\text{-}in}}$$

$$C_{wall} \frac{dT_{wall}}{dt} = \frac{T_{air} - T_{wall}}{R_{wall\text{-}in}} + \frac{T_{out} - T_{wall}}{R_{wall\text{-}out}}$$

Each equation is no more complicated than what we have solved, but there are three of them and they're coupled.

First, let's come up with some abbreviations:

$C_{mass} = C_m$ | $T_{mass} = T_m$ | $R_{mass\text{-}air} = R_{ma}$
$C_{air} = C_a$ | $T_{air} = T_a$ | $R_{wall\text{-}in} = R_{wi}$
$C_{wall} = C_w$ | $T_{wall} = T_w$ | $R_{wall\text{-}out} = R_{wo}$
 | $T_{out} = T_o$ |

4. Now, let's re-arrange the equations

$$\frac{d}{dt}\begin{bmatrix} T_m \\ T_a \\ T_w \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_m R_{ma}} & +\frac{1}{C_m R_{ma}} & 0 \\ \frac{1}{C_a R_{ma}} & -\left(\frac{1}{C_a R_{ma}} + \frac{1}{C_a R_{wi}}\right) & \frac{1}{C_a R_{wi}} \\ 0 & \frac{1}{C_w R_{wi}} & -\left(\frac{1}{C_w R_{wi}} + \frac{1}{C_w R_{wo}}\right) \end{bmatrix} \begin{bmatrix} T_m \\ T_a \\ T_w \end{bmatrix}$$

$$+ \begin{bmatrix} 1/C_m \\ 0 \\ 0 \end{bmatrix} q_{solar} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{C_w R_{wo}} \end{bmatrix} T_o$$

c. Vector form

$$\bar{T} = \begin{bmatrix} T_m \\ T_a \\ T_w \end{bmatrix}$$

$$\bar{\bar{A}} = \begin{bmatrix} \text{3x3 matrix} \\ \text{of R's and C's} \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} 1/C_m \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{C_w R_{wo}} \end{bmatrix}$$

$$\frac{d}{dt}\bar{T} = \bar{\bar{A}}\,\bar{T} + \bar{B}\,q_{solar} + \bar{C}\,T_o$$

6. Now (the magic) let's guess a solution - at least for a simple problem.

Consider the so-called homogeneous problem, where $q_{solar} = 0$ and $T_o = 0$. (In effect, we only insist that both are constant)

Recall
$$\frac{dT_{in}}{dt} + \frac{1}{\tau} T_{in} = 0$$
$$T_{in}\Big|_{t=0} = T_{in,initial}$$

Solution:
$$T_{in} = T_{in,initial}\, e^{-t/\tau}$$

Remember, too, that $\tau = RC$

For the vector problem, take a big leap and try
$$\bar{T} = e^{\bar{\bar{A}}t}\,\bar{T}_{initial}$$

(Note the order on the right - it matters with vectors)

This indeed is the correct solution. But what is

$$e^{\bar{\bar{A}}t} \quad ??$$

It has a name: the matrix exponential

It can be defined by its Taylor Series expansion

$$e^{\bar{\bar{A}}t} = \bar{\bar{I}} + \bar{\bar{A}}t + \bar{\bar{A}} \cdot \bar{\bar{A}} \frac{t^2}{2} + \ldots$$

We could go on to deal with q_{solar} and T_{out} but will not. The important point: we can tailor a model to account for what we think is important, physically, and then easily get an answer. The math software Matlab, for example, is designed to solve just this kind of problem. You provide $\bar{\bar{A}}$, $\bar{B}$ and $\bar{C}$, plus initial conditions, and it produces an answer.

Historical note. Where did this approach come from? Not buildings! The 1960's Apollo program, in which Draper Lab was a key participant.

$$\dot{\bar{x}} = \bar{\bar{A}}\bar{x} + \bar{B}u$$

$$\bar{x}(t) = e^{\bar{\bar{A}}t}\,\bar{x}(0) + \int_0^t e^{\bar{\bar{A}}(t-\eta)}\bar{B}u(\eta)\,d\eta$$

$$\dot{\bar{T}} = \bar{\bar{A}}x + \bar{B}q$$

If q is constant

$$T(t) = e^{A(t)}\,T(0) + \int_0^t e^{A(t-\eta)}\bar{B}q(\eta)\,d\eta$$

$$= e^{A(t)}\,T(0) + \left[\int_0^t e^{A(t-\eta)}\,d\eta\right]\bar{B}q$$

$$= e^{A(t)}\,T(0) + \left[\int_0^t e^{A(t-\eta)}A\,d\eta\right]\bar{\bar{A}}^{-1}\bar{B}\,q$$

$$= e^{\bar{\bar{A}}(t)}\,\bar{T}(0) + e^{\bar{\bar{A}}t}\cdot(\bar{\bar{I}} - e^{-\bar{\bar{A}}t})\,\bar{\bar{A}}^{-1}\bar{B}\,q$$

$$= e^{\bar{\bar{A}}(t)}\,\bar{T}(0) - (\bar{\bar{I}} - e^{\bar{\bar{A}}t})\,\bar{\bar{A}}^{-1}\bar{B}q$$

Scalar case $A(t) = -\frac{1}{\tau}$ $\quad B = \frac{1}{C}$

$$e^{-t/\tau}\,T(0) + (1 - e^{-t/\tau})Rq \quad ✓$$

2.2. Compute walls parameters for each zone

- i : index of zone
- j : index for adjacent zone
- m : index for wall in zone i
- n : index for layer in wall m. Numbering of layers from 1 to N from outdoor to indoor or from adjacent room to room considered.

2.2.1. External wall

Wall m is external if in connection with any surroundings. Represent the wall by a 1st order model (figure 2).

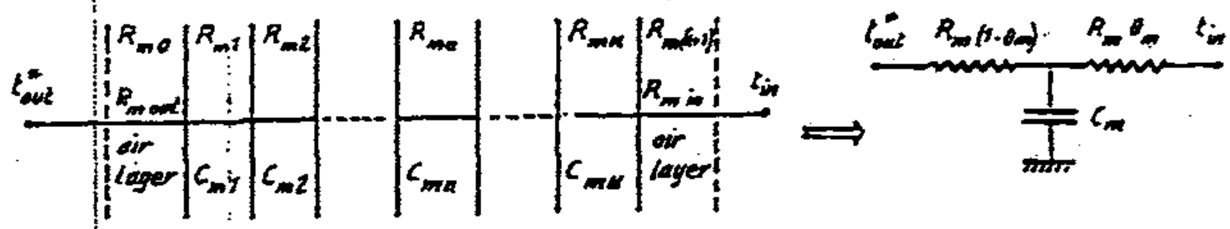


Figure 2 : 1st order model for external wall.

a. . For each layer :

outdoor air layer resistance : $R_{m0} = R_{m\,out} = \dfrac{1}{A_m\, h_{out\,m}}$

layer n resistance : $R_{mn} = \dfrac{e_{mn}}{A_m\, k_{mn}}$

layer n capacitance : $C_{mn} = \rho_{mn}\, A_m\, e_{mn}\, c_{p\,mn}$

indoor air layer resistance : $R_{m(N+1)} = R_{m\,in} = \dfrac{1}{A_m\,h_{in\,m}}$

Generally, all layers of wall m present the same surface. It is then easier to compute layers and walls parameters per unit area and introduce areas only when the zone models are constituted.

b. . <u>Overall parameters of wall m</u>

Overall resistance : $R_m = \sum_{n=0}^{N+1} R_{mn}$

Overall capacitance : $C_m = \sum_{n=1}^{N} C_{mn}$

Accessibility of capacitance :

$$\Theta_m = 1 - \sum_{n=1}^{N} \frac{\left\{ \sum_{p=0}^{n-1} R_{mp} + \frac{R_{mn}}{2} \right\} C_{mn}}{R_m\,C_m}$$

Time constant : $\tau_m = \Theta_m(1-\Theta_m)R_m\,C_m$

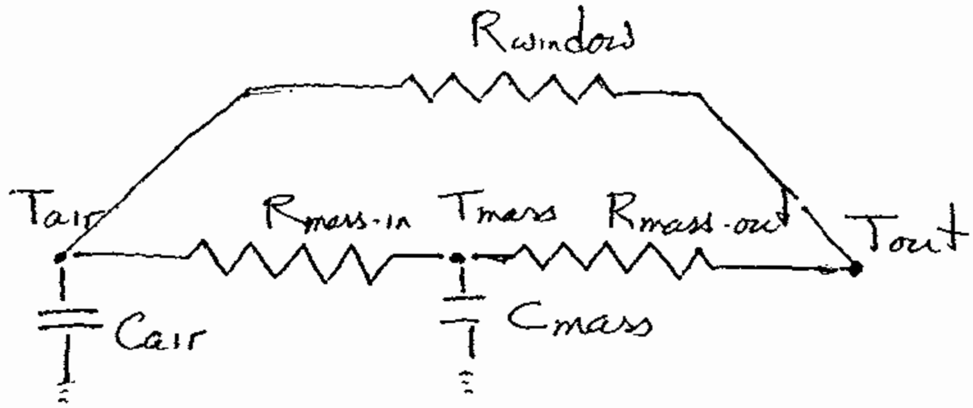
2.2.2. Internal wall

Wall m is internal if both sides of it have the same temperature. It may be a wall completely included in a zone or a wall separating a simulated zone and a nearby zone which is not simulated but whose temperature is assumed to vary in a strict similarity with the simulated zone's temperature.

a. . First represent the wall by a first order model (see procedure for external wall). Thus compute R_m, C_m, Θ_m.

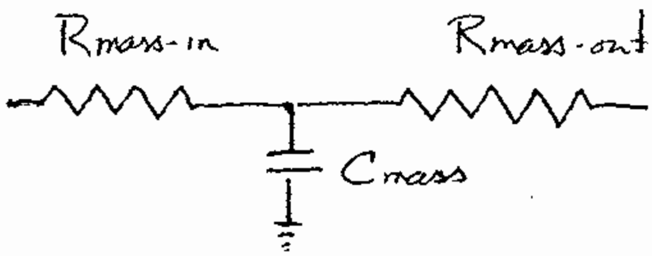
b. . Define 2d order model (figure 3.)

Here's one more model, often used in buildings research:

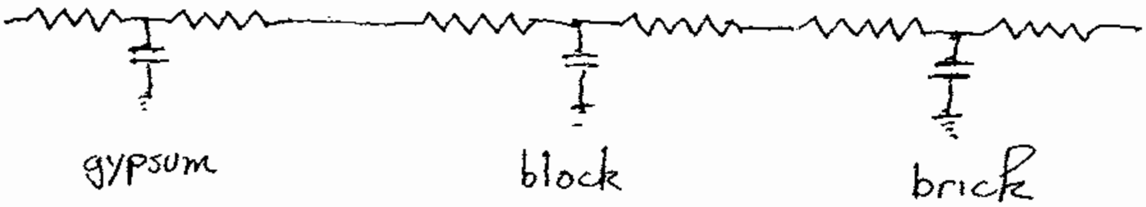


Cair includes air and lightweight stuff: furniture, lamps, trash cans

Final point: Is the temperature constant throughout a massive wall? Of course not! What then?



is fine, but the values for the parameters should be derived from something like



Working with Matlab

Our format of the multi-temperature model

$$\dot{\bar{T}} = \bar{\bar{A}}\,\bar{T} + \bar{B}\, q_{solar} + \bar{C}\, T_{out}$$

Matlab's format for "state-space" models

$$\dot{\bar{x}} = \bar{\bar{A}}\,\bar{x} + \bar{\bar{B}}\,\bar{u}$$

$$\bar{y} = \bar{\bar{C}}\,\bar{x} + \bar{\bar{D}}\,\bar{u}$$

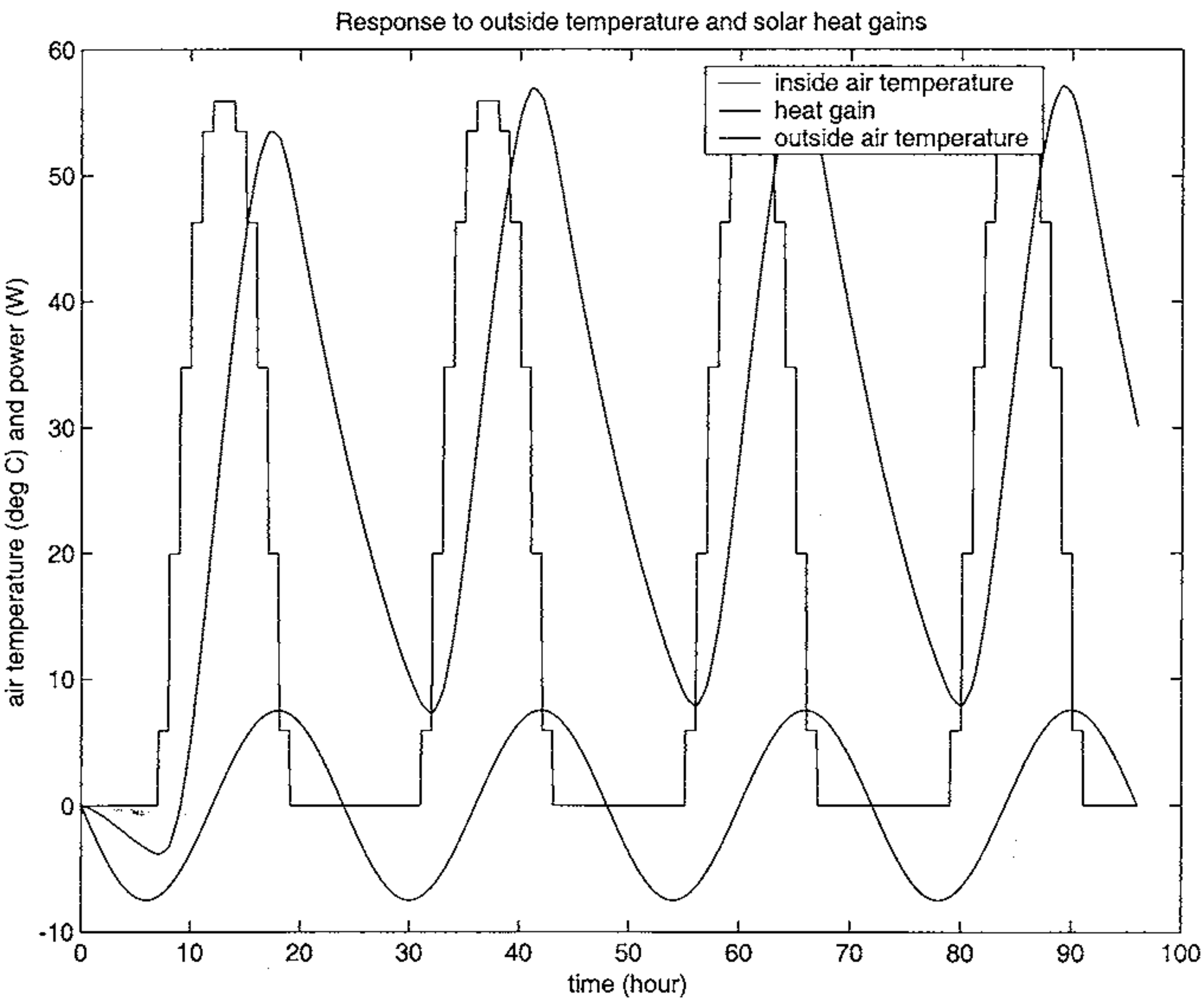
where $\bar{u}$ is a control vector and $\bar{y}$ is an output vector

To use Matlab, all inputs from q_{solar} and T_{out} must be coupled to the Matlab $\bar{\bar{B}}$ matrix.

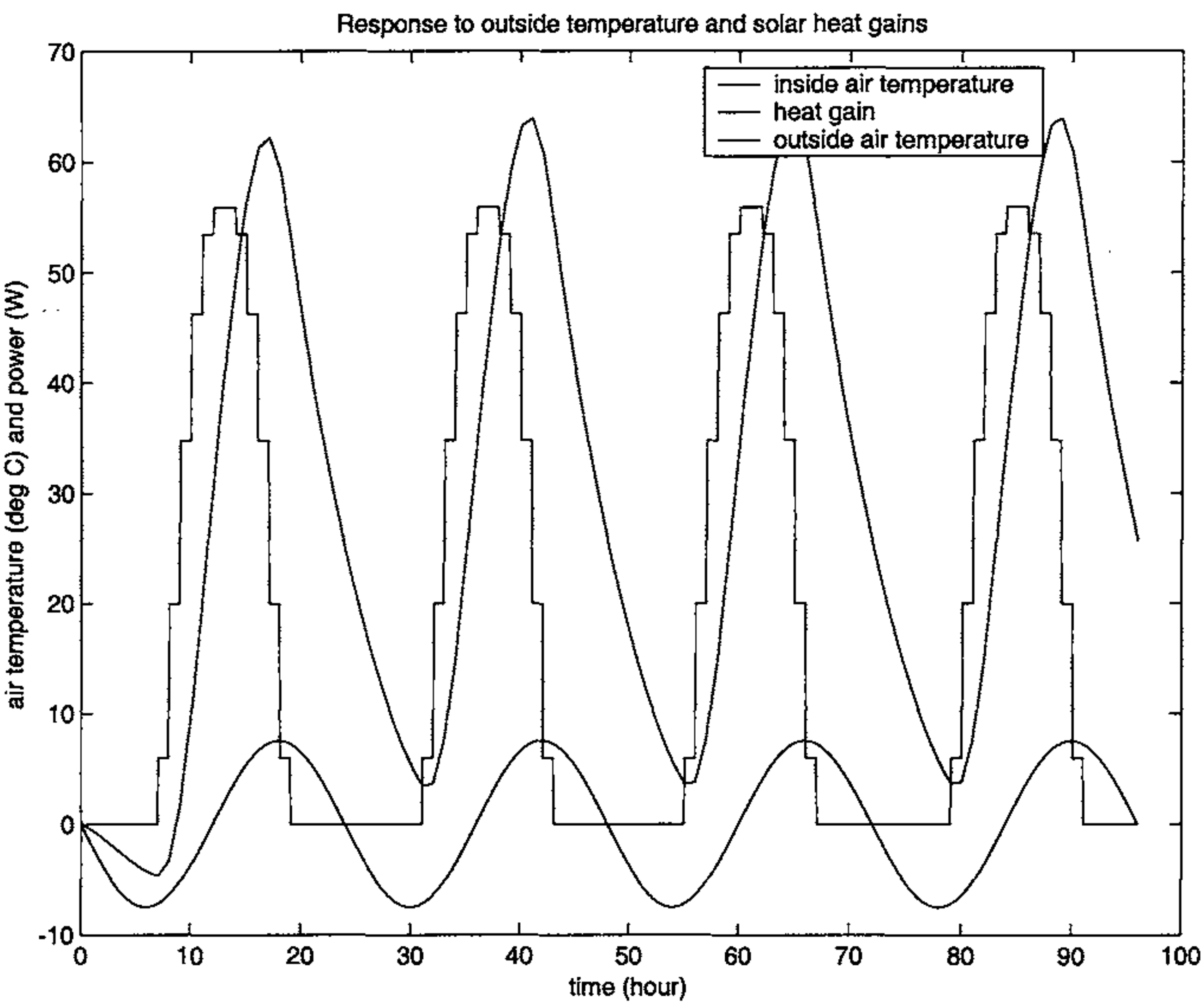
We will define $\bar{\bar{C}}$ as a row vector to select a single temperature from the three temperatures we are modeling.

We will always set $\bar{\bar{D}}$ to zero.

R_{wo} 1.9
R_{wi} 0.1

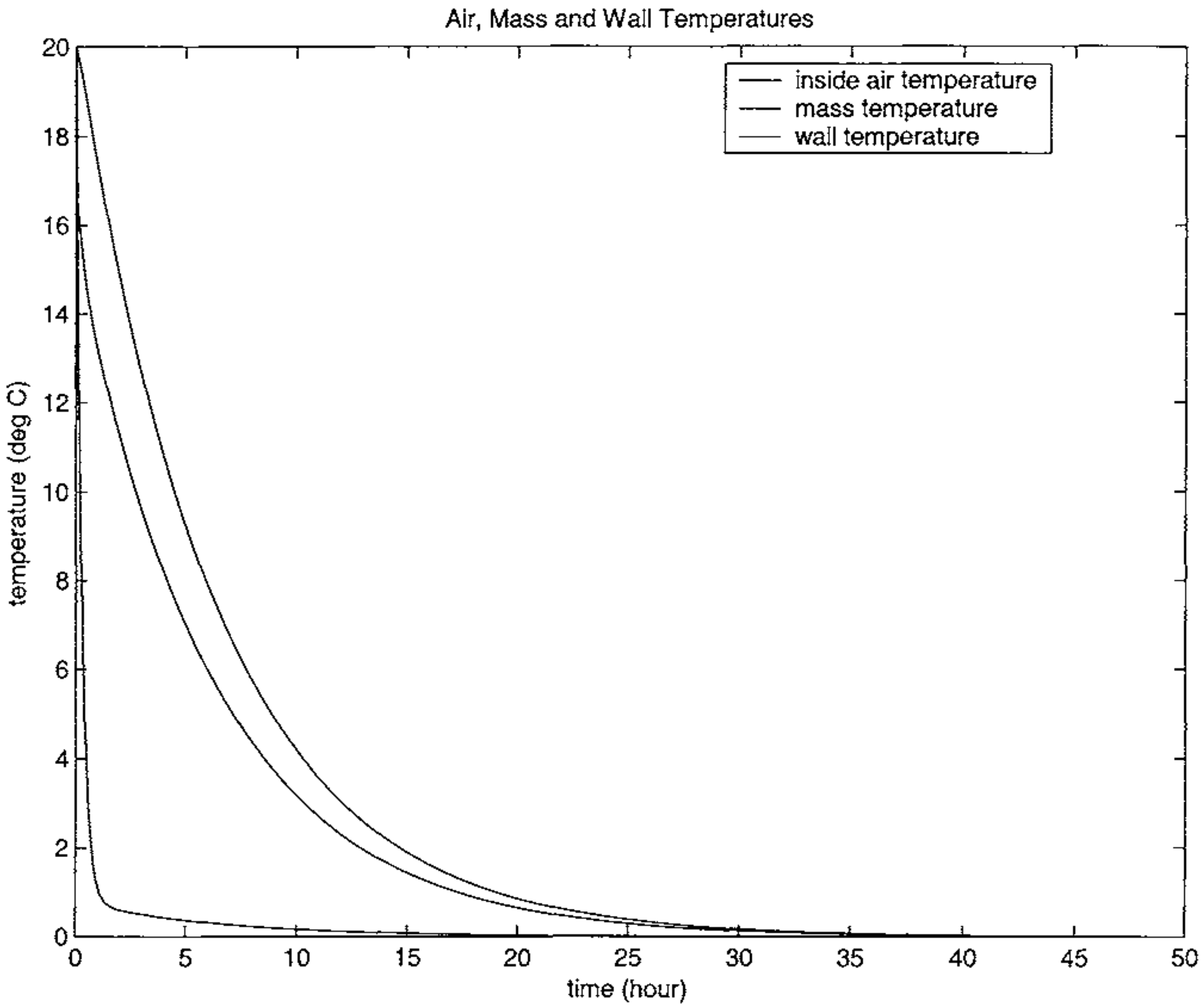

Response to outside temperature and solar heat gains
inside air temperature
heat gain
outside air temperature
air temperature (deg C) and power (W)
time (hour)
60
50
40
30
20
10
0
-10
0
10
20
30
40
50
60
70
80
90
100

Rwo 0.1
Rwi 1.9



Response to outside temperature and solar heat gains
inside air temperature
heat gain
outside air temperature
air temperature (deg C) and power (W)
time (hour)
70
60
50
40
30
20
10
0
-10
0
10
20
30
40
50
60
70
80
90
100

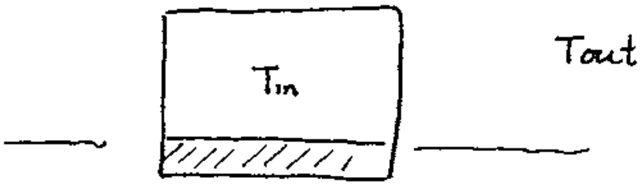
Rwd 0.1
Rwc 1.9


Air, Mass and Wall Temperatures
inside air temperature
mass temperature
wall temperature
temperature (deg C)
time (hour)
0
5
10
15
20
25
30
35
40
45
50
0
2
4
6
8
10
12
14
16
18
20

11/19/96

House model

Frequency Analyses



1R 1C

$T_{in} = T_{mass}$

Thermal resistance $R = \frac{1}{UA}$ for building envelope

$$C \frac{dT_{in}}{dt} = \frac{1}{R}(T_o - T_{in})$$

$$RC = \tau$$

$$\tau \frac{dT_{in}}{dt} = T_o - T_{in}$$

→ Take Laplace transform

$$(\tau s + 1)\, T_{in}(s) = T_o(s)$$

$$T_{in}(s) = \frac{1}{\tau s + 1} T_o(s) = G(s)\, T_o(s)$$

→ Solve the ode for $T_o = e^{j\omega t}$

$$\frac{dT_{in}}{dt} + \frac{1}{\tau} T_{in} = \frac{1}{\tau} e^{j\omega t}$$

Use an integrating factor

$$e^{t/\tau}\left(\frac{dT_{in}}{dt} + \frac{1}{\tau}T_{in}\right) = \frac{e^{(j\omega + 1/\tau)t}}{\tau}$$

$$\frac{d}{dt}\left(e^{t/\tau}T_{in}\right) = \frac{e^{(j\omega+1/\tau)t}}{\tau}$$

$$e^{t/\tau}T_{in} = \frac{1}{\tau}\int^{t} e^{(j\omega+1/\tau)t'}\,dt'$$

$$= \frac{1}{\tau(j\omega+1/\tau)}e^{(j\omega+1/\tau)t} + C$$

$$T_{in}(t) = \frac{1}{(j\omega\tau+1)}e^{j\omega t} + Ce^{-t/\tau}$$

Pick $C = T_{in}(t=0)$ to satisfy initial condition

→ Note that initial condition will decay for stable system

→ At steady state

$$T_{in}(t) = \frac{1}{(j\omega\tau+1)}e^{j\omega t}$$

$$= G(j\omega)\,e^{j\omega t} \quad *$$

Note that in this case, as in general, the response is governed by the transfer function, with $s = j\omega$.

For the house,

$$G(j\omega) = \frac{1}{j\omega\tau + 1}$$

$$= \frac{-j\omega\tau + 1}{\omega^2\tau^2 + 1}$$

$$= \frac{1}{\omega^2\tau^2+1} - j\frac{\omega\tau}{\omega^2\tau^2+1}$$

$$= A(\omega)\, e^{j\phi(\omega)}$$

where $A(\omega) = \sqrt{Re^2 + Im^2}$

$$= \frac{\sqrt{1+\omega^2\tau^2}}{1+\omega^2\tau^2}$$

$$= \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

$$\phi(\omega) = \tan^{-1}(-\omega\tau)$$

→ Consider $A(\omega)$

low ω or very small τ

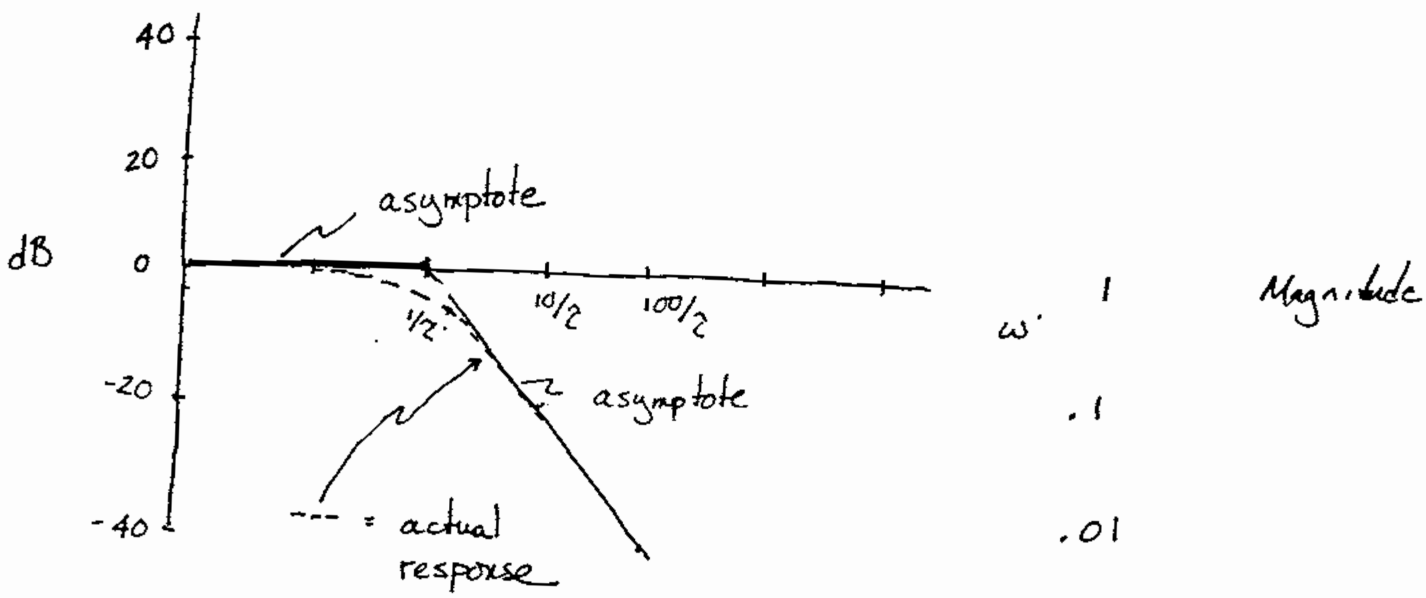
For $\omega \ll 1/\tau$, $A(\omega) \cong 1$ $\quad 20 \log A = 0$

$\omega = 1/\tau$ $\quad A(\omega) = \frac{1}{\sqrt{2}} = \frac{1}{1.4}$ $\quad 20 \log A \cong -3 \text{ dB}$

$\omega \gg 1/\tau$ $\quad A(\omega) \cong \frac{1}{\omega\tau}$ $\quad 20 \log A = -20 \log \omega\tau$

$$-20 \log \omega \tau = -20(\log \omega + \log \tau)$$

When $-20 \log \omega \tau$ is plotted against $\log \omega$, we obtain a slope of -20 dB/decade and an intercept of 0 dB at $\omega = 1/\tau$

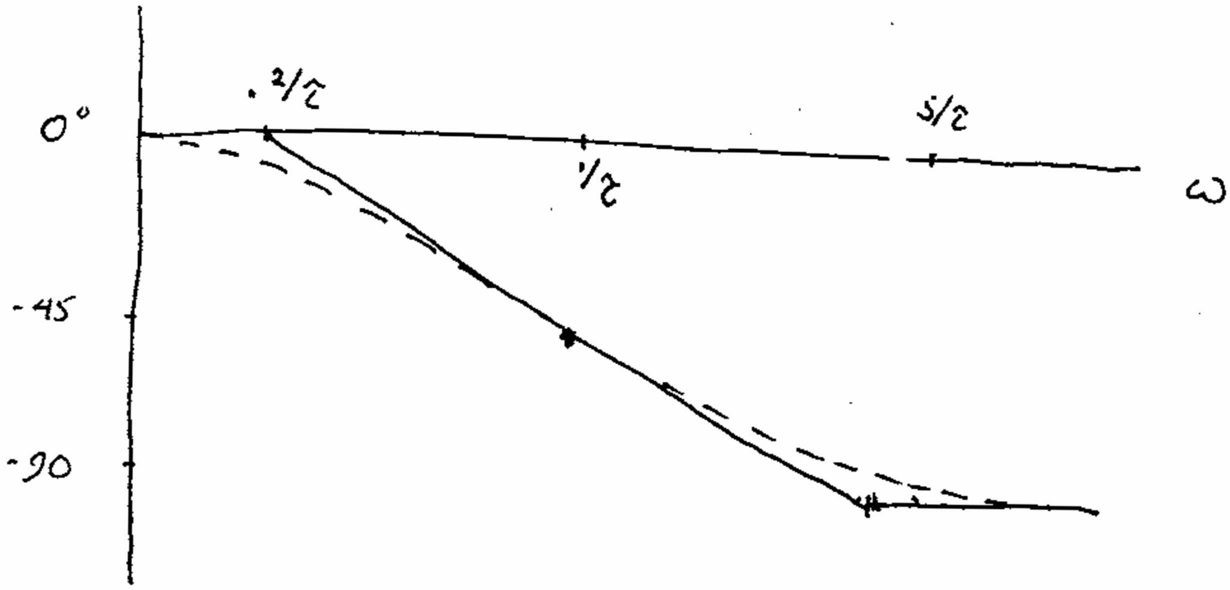


→ Now consider phase angle $\phi = \tan^{-1}(-\omega \tau)$

$\omega \ll 1/\tau \quad \phi = 0$

$\omega = 1/\tau \quad \phi = -45°$

$\omega \gg 1/\tau \quad \phi = -90°$



→ What does this tell us about our house?

Maximum phase shift is 90°

360° => 24 hours   90° => 6 hours

If Tout peaks at 3 pm, our house will peak no later than 9 pm

Let Tout have a period of 2π radians (or 360°) in 24 hours, or ~ 1/4 radian/hour

If $\tau = 10$ hours, the amplitude in T_{in} is reduced to about .37 × the amplitude of Tout

If $\tau = 40$ hours, the amplitude is reduced to 0.1 the amplitude of Tout