Numerical Methods for PDEs

Integral Equation Methods, Lecture 3
Discretization Convergence Theory

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Outline

Integral Equation Methods

- Reminder about Galerkin and Collocation

Example of convergence issues in 1D

- First and second kind integral equations
- Develop some intuition about the difficulties

Convergence for second kind equations

- Consistency and stability issues

Nystrom Methods

- High order convergence

Integral Equation Basics

Basis Function Approach

Basic Idea

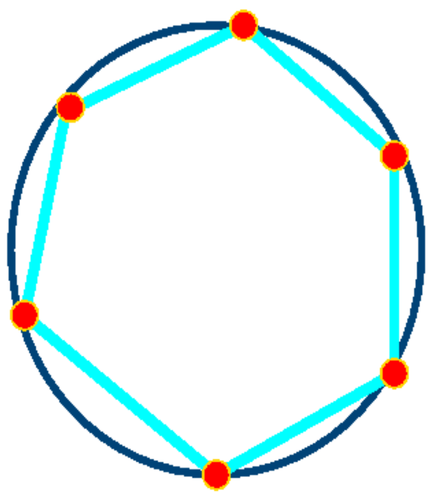
Integral equation: $\Psi(x) = \int G(x, x')\sigma(x')dS'$

Represent $\sigma_n(x) = \sum_{i=1}^{n} \sigma_{ni} \underbrace{\varphi_i(x)}_{Basis\ functions}$

Example Basis

Represent circle with straight lines

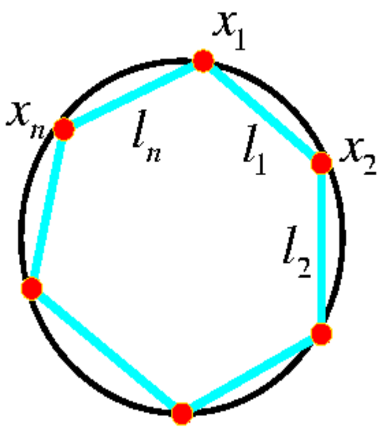
Assume σ is constant along each line



Integral Equation Basics

Basis Function Approach

Piecewise Constant Straight Sections Example



1) Pick a set of n Points on the surface
2) Define a new surface by connecting points with n lines.
3) Define $\varphi_i(x) = 1$ if x is on line l_i otherwise, $\varphi_i(x) = 0$

$$\Psi(x) = \int_{\substack{approx\\surface}} G(x,x') \sum_{i=1}^{n} \sigma_{ni} \varphi_i(x') dS' = \sum_{i=1}^{n} \sigma_{ni} \int_{line\, l_i} G(x,x') dS'$$

How do we determine the σ_{ni} 's?

Integral Equation Basics

Basis Function Approach

Residual Definition and Minimization

$$R(x) \equiv \Psi(x) - \int_{\substack{\text{approx}\\\text{surface}}} G(x, x') \sum_{i=1}^{n} \sigma_{ni} \varphi_i(x') dS'$$

We will pick the σ_{ni} 's to make $R(x)$ small.

General approach: Pick a set of test functions $\phi_1, \ldots, \phi_n$, and force $R(x)$ to be orthogonal to the set;

$$\int \phi_i(x) R(x) dS = 0 \quad for\ all\ i$$

Integral Equation Basics

Basis Function Approach

Residual Minimization Using Test Functions

$$\boxed{\int \phi_i(x) R(x) dS = 0} \Rightarrow$$

$$\int \phi_i(x)\Psi(x)dS - \int \int_{\substack{\text{approx}\\\text{surface}}} \phi_i(x) G(x,x') \sum_{j=1}^{n} \sigma_{nj}\varphi_j(x') dS' dS = 0$$

We will generate different methods by choosing the $\phi_1, \ldots, \phi_n$

Collocation : $\phi_i(x) = \delta(x - x_{t_i})$ (point matching)

Galerkin Method : $\phi_i(x) = \varphi_i(x)$ (basis = test)

Weighted Residual Method : $\phi_i(x) = 1 \; if \; \varphi_i(x) \neq 0$ (averages)

Integral Equation Basics

Basis Function Approach

Collocation

Collocation: $\phi_i(x) = \delta(x - x_{t_i})$ (point matching)

$\int \delta(x - x_{t_i}) R(x) dS = R(x_{t_i}) = 0 \Rightarrow$

$$\sum_{j=1}^{n} \sigma_{nj} \overbrace{\int_{\substack{\text{approx} \\ \text{surface}}} G(x_{t_i}, x') \varphi_j(x') dS'}^{A_{i,j}} = \Psi(x_{t_i})$$

$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \sigma_{n1} \\ \vdots \\ \vdots \\ \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \Psi(x_{t_1}) \\ \vdots \\ \vdots \\ \Psi(x_{t_n}) \end{bmatrix}$$

Integral Equation Basics

Basis Function Approach

Galerkin

Galerkin: $\phi_i(x) = \varphi_i(x)$ (test=basis)

$$\int \varphi_i(x)\, R(x)\, dS = \int \varphi_i(x)\, \Psi(x)\, dS - \int_{\substack{approx\\surface}} \int \varphi_i(x) G(x,x') \sum_{j=1}^{n} \sigma_{nj} \varphi_j(x')\, dS' dS = 0$$

$$\underbrace{\int_{\substack{approx\\surface}} \varphi_i(x') \Psi(x)\, dS'}_{b_i} = \sum_{j=1}^{n} \sigma_{nj} \underbrace{\int_{\substack{approx\\surface}} \int_{\substack{approx\\surface}} G(x,x') \varphi_i(x) \varphi_j(x')\, dS' dS}_{A_{i,j}}$$

$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \sigma_{n1} \\ \vdots \\ \vdots \\ \sigma_{nn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

If $G(x,x') = G(x',x)$ then $A_{i,j} = A_{j,i} \Rightarrow$ **A is symmetric**

Convergence Analysis

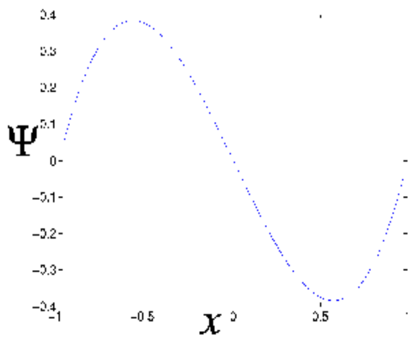
Example Problems

1D First Kind Equation

$$\Psi(x) = \int_{-1}^{1} |x - x'| \sigma(x') dS' \quad x \in [-1, 1]$$

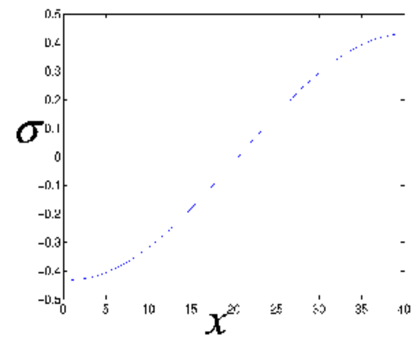
The potential is given

$$\Psi(x) = x^3 - x$$



The density must be computed

$\sigma(x)$ is unknown



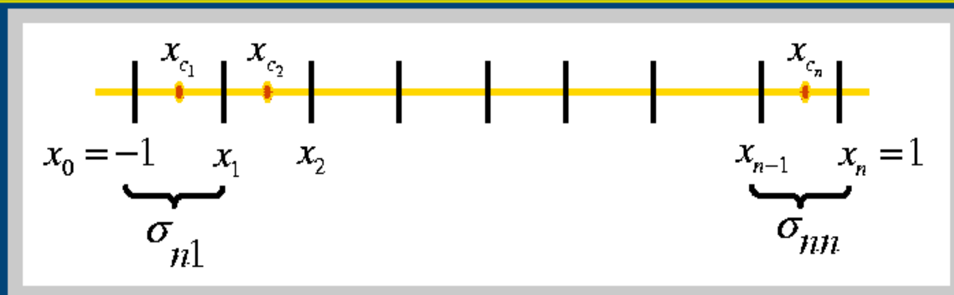
Convergence Analysis

Example Problems

Collocation Discretization of 1D Equation

$$\Psi(x) = \int_{-1}^{1} |x - x'| \sigma(x') dS' \quad x \in [-1, 1]$$

Centroid Collocated Piecewise Constant Scheme



$$\Psi(x_{c_i}) = \sum_{j=1}^{n} \sigma_{nj} \int_{x_{j-1}}^{x_j} |x_{c_i} - x'| dS'$$

Convergence Analysis

Example Problems

Collocation Discretization of 1D Equation-The Matrix

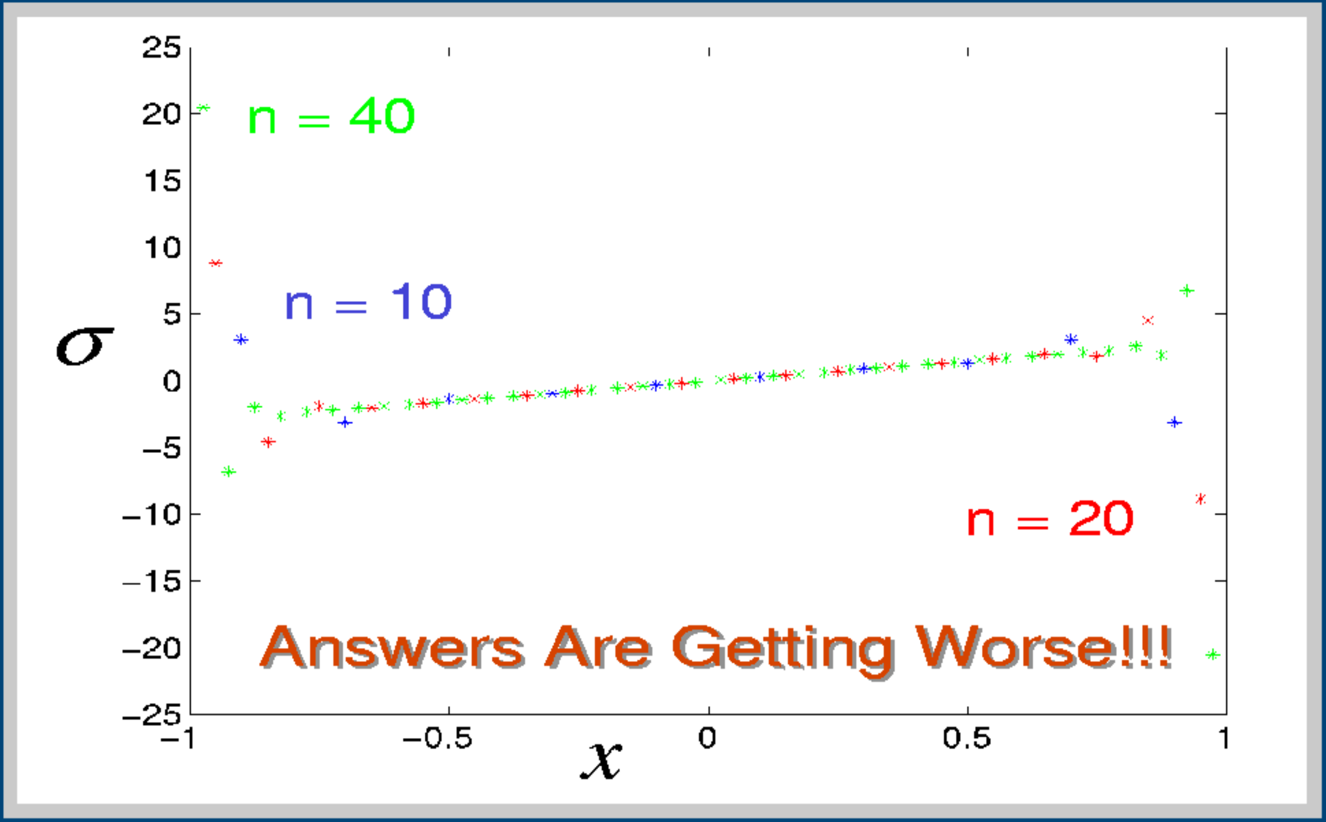
One column for each density value

$$\begin{bmatrix} \int_{x_0}^{x_1} \left| x_{c_1} - x' \right| dS' & \cdots & \int_{x_{n-1}}^{x_n} \left| x_{c_1} - x' \right| dS' \\ \vdots & \ddots & \vdots \\ \int_{x_0}^{x_1} \left| x_{c_n} - x' \right| dS' & \cdots & \int_{x_{n-1}}^{x_n} \left| x_{c_n} - x' \right| dS' \end{bmatrix} \begin{bmatrix} \sigma_{n1} \\ \vdots \\ \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \Psi\left(x_{c_1}\right) \\ \vdots \\ \Psi\left(x_{c_n}\right) \end{bmatrix}$$

One row for each collocation point

Convergence Analysis

Numerical Results with Increasing n



Convergence Analysis

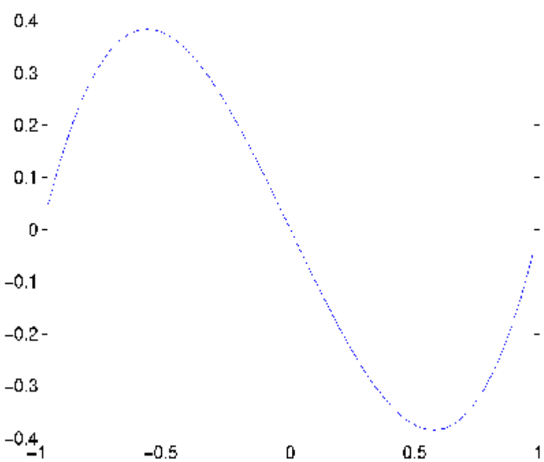
Example Problems

1D Second Kind Equation

$$\Psi(x) = \sigma(x) + \int_{-1}^{1} |x - x'| \sigma(x') dS' \quad x \in [-1,1]$$

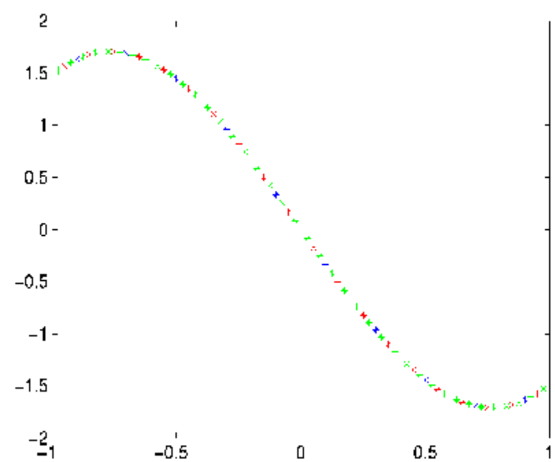
The potential is given

$$\Psi(x) = x^3 - x$$



The density must be computed

$\sigma(x)$ is unknown



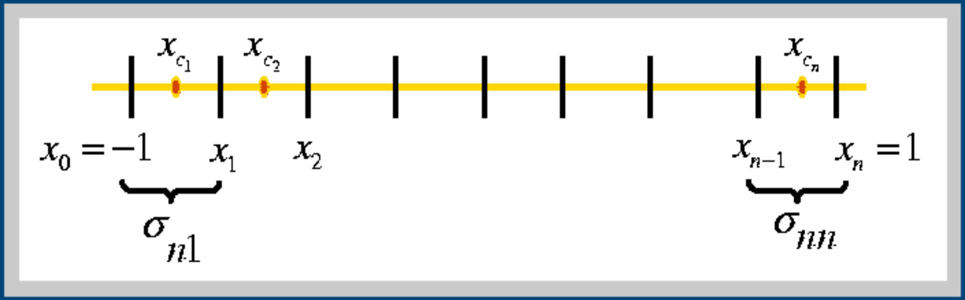
Convergence Analysis

Example Problems

Collocation Discretization of 1D Equation

$$\Psi(x) = \sigma(x) + \int_{-1}^{1} |x - x'| \sigma(x') dS' \quad x \in [-1, 1]$$

Centroid Collocated Piecewise Constant Scheme



$$\Psi(x_{c_i}) = \sigma_{ni} + \sum_{j=1}^{n} \sigma_{nj} \int_{x_{j-1}}^{x_j} |x_{c_i} - x'| dS'$$

Convergence Analysis

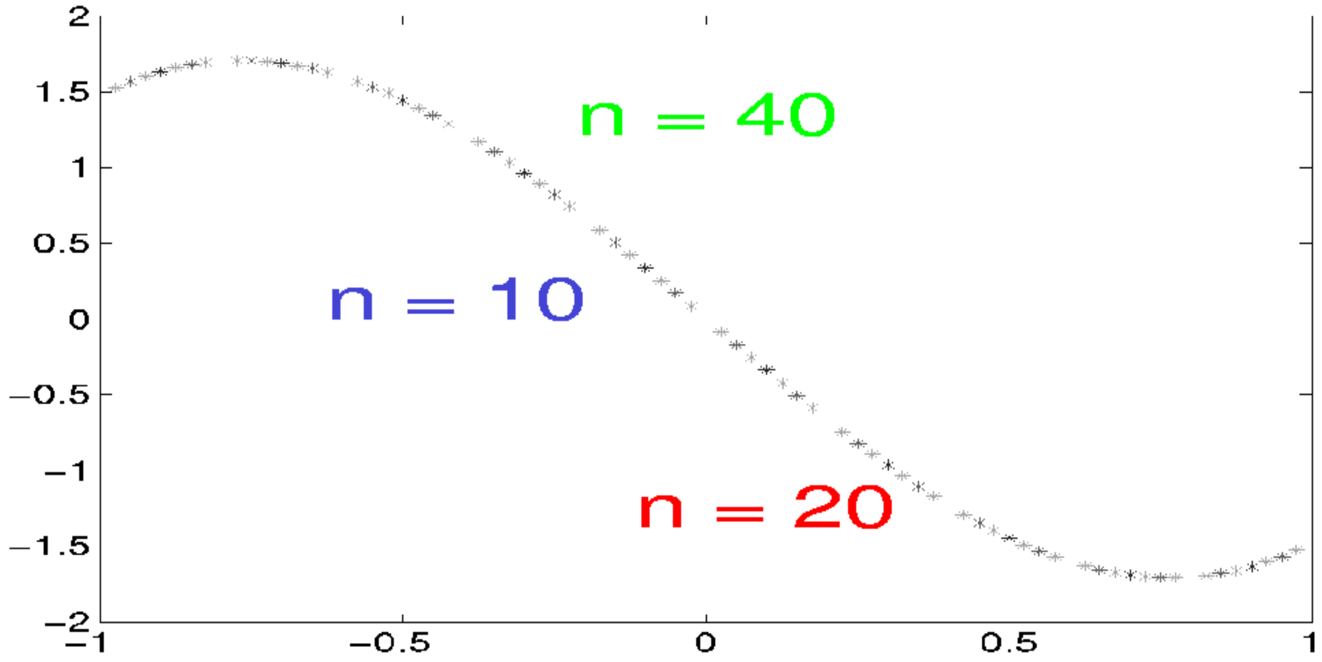
Example Problems

Collocation Discretization of 1D Equation-The Matrix

$$\begin{bmatrix} 1+\int_{x_0}^{x_1}\left|x_{c_1}-x'\right|dS' & \cdots & \int_{x_{n-1}}^{x_n}\left|x_{c_1}-x'\right|dS' \\ \vdots & \ddots & \vdots \\ \int_{x_0}^{x_1}\left|x_{c_n}-x'\right|dS' & \cdots & 1+\int_{x_{n-1}}^{x_n}\left|x_{c_n}-x'\right|dS' \end{bmatrix}\begin{bmatrix}\sigma_{n1}\\ \vdots \\ \sigma_{nn}\end{bmatrix}=\begin{bmatrix}\Psi\left(x_{c_1}\right)\\ \vdots \\ \Psi\left(x_{c_n}\right)\end{bmatrix}$$

Convergence Analysis

Numerical Results with Increasing n



Answers Are Improving!!!

Convergence Analysis

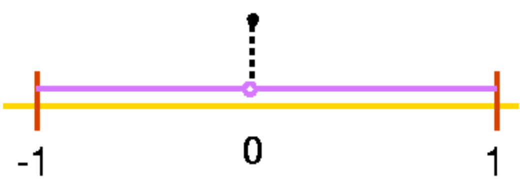
Example Problems

1D First Kind Equation Difficulty

Denote the integral operator as K

$$K\sigma \equiv \int_{-1}^{1} |x - x'|\sigma(x')dS' \Rightarrow K\sigma = \Psi$$

The integral operator is **singular** : K **has a null space**



$$\sigma_0(x) = 0,\ x \neq 0,\ \ \sigma_0(0) = 1$$

$$K\sigma_0 = \int_{-1}^{1} |x - x'|\sigma_0(x')dS' = 0$$

If $K\sigma^a = \Psi \quad then \quad K(\sigma^a + \sigma_0) = \Psi$

Convergence Analysis

Example Problems

1D First Kind Equation Difficulty from the Matrix

Collocation generates a discrete form of K

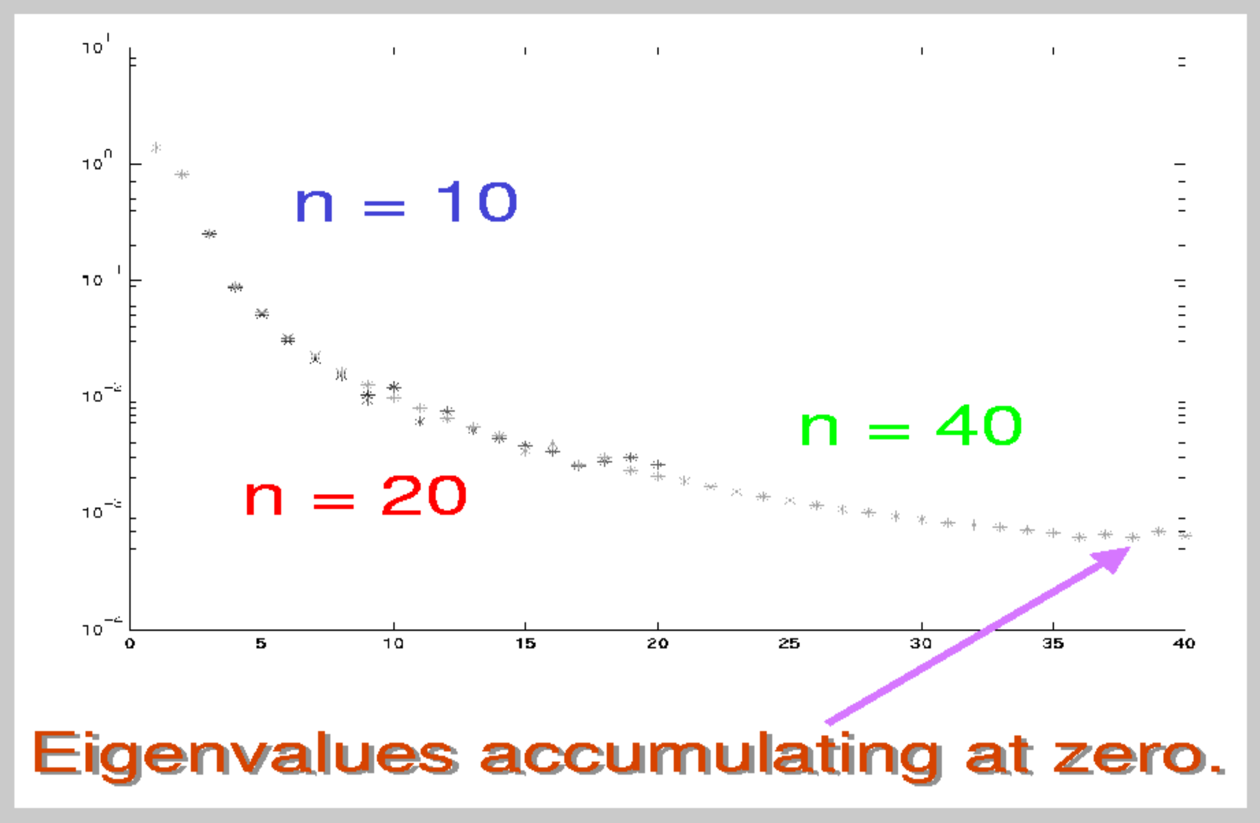
$$K\sigma = \Psi \quad \rightarrow K_n \sigma_n = \Psi_n$$

$$\underbrace{\begin{bmatrix} \int_{x_0}^{x_1} \left| x_{c_1} - x' \right| dS' & \cdots & \int_{x_{n-1}}^{x_n} \left| x_{c_1} - x' \right| dS' \\ \vdots & \ddots & \vdots \\ \int_{x_0}^{x_1} \left| x_{c_n} - x' \right| dS' & \cdots & \int_{x_{n-1}}^{x_n} \left| x_{c_n} - x' \right| dS' \end{bmatrix}}_{\underline{K}_n} \begin{bmatrix} \sigma_{n1} \\ \vdots \\ \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \Psi\left(x_{c_1}\right) \\ \vdots \\ \Psi\left(x_{c_n}\right) \end{bmatrix}$$

The matrix $\underline{K}_n$ is the not the operator K_n!

Convergence Analysis

Numerical Results with Increasing n

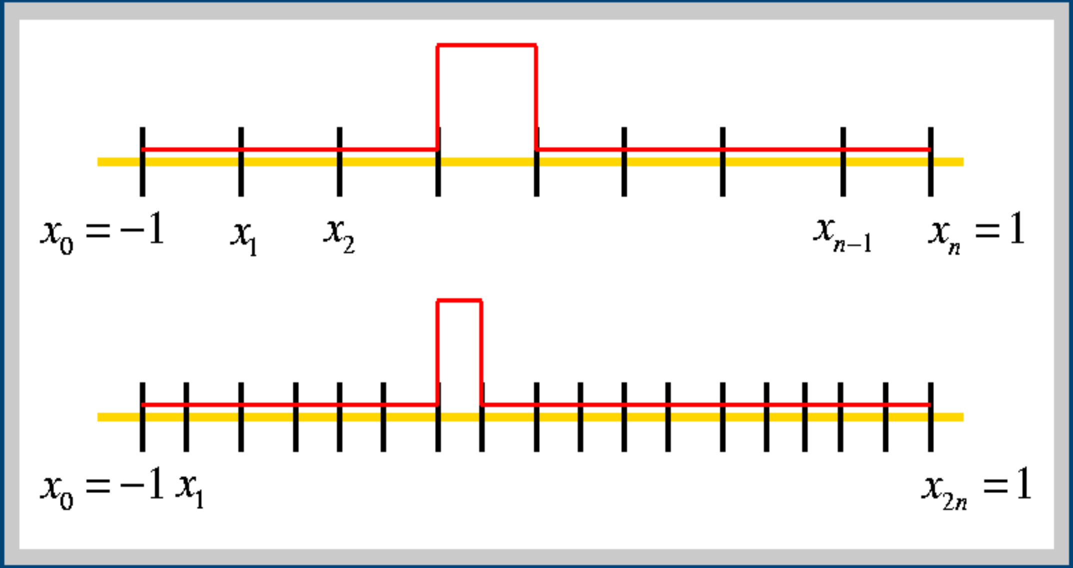


Convergence Analysis

Example Problems

Intuition About the Eigenvalues

As the discretization is refined, $\sigma_0(x)$ becomes better approximated



As the discretization is refined, K's null space can be more accurately represented.

Convergence Analysis

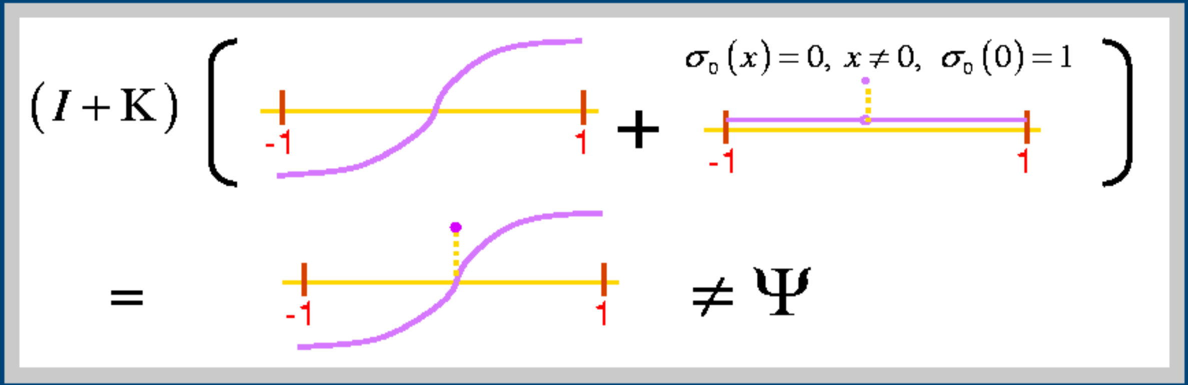
Example Problems

Second kind Equation has Fewer Problems

Second Kind equation

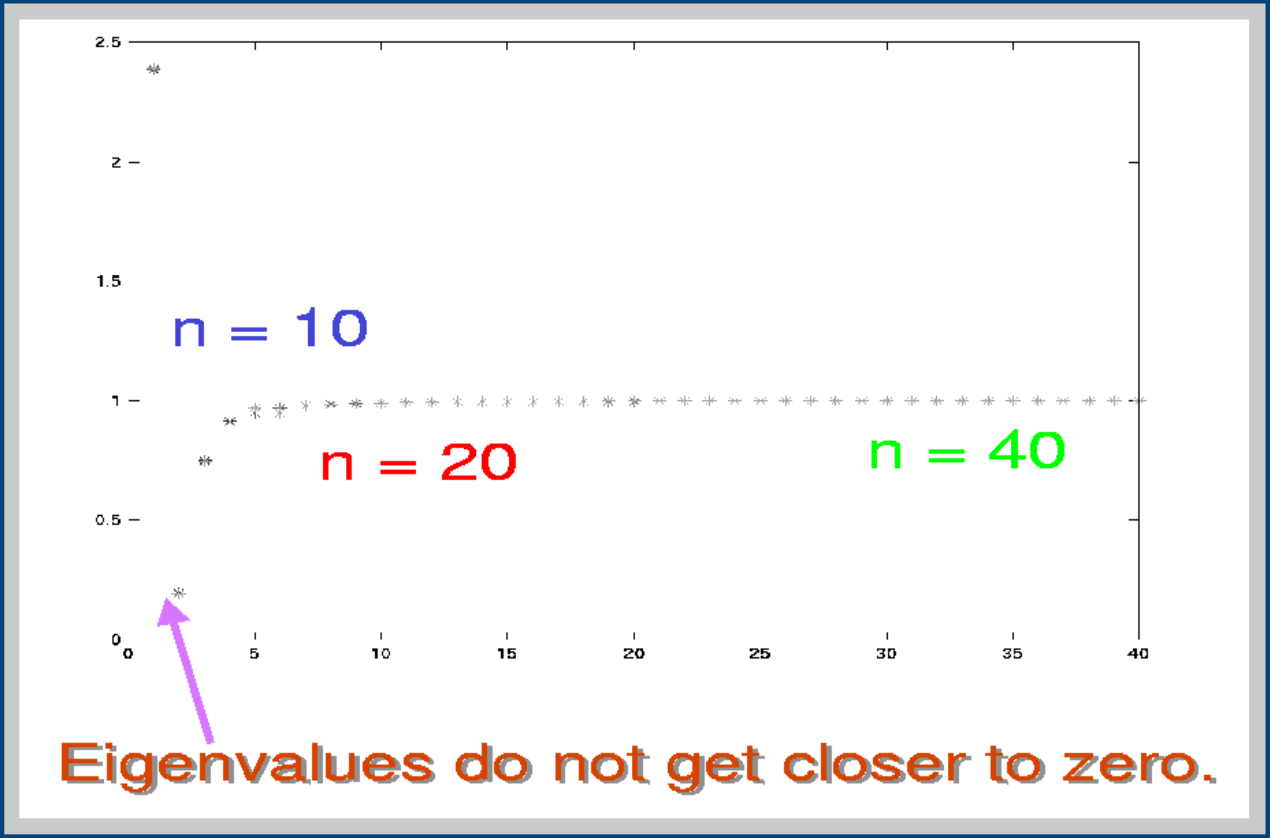
$$(I+K)\sigma \equiv \sigma(x) + \int_{-1}^{1} |x - x'|\sigma(x')dS' \Rightarrow (I+K)\sigma = \Psi$$

$$(I+K)(\sigma_0 + \sigma) \neq (I+K)\sigma$$



Convergence Analysis

Numerical Results with Increasing n



Convergence Analysis

Second Kind Theory

General Framework

General Second kind integral equation

$$\Psi(x) = \sigma(x) + \int G(x, x')\sigma(x')dx' \Rightarrow \Psi = (I + K)\sigma$$

Discrete equivalent

$$\Psi_n = (I + K_n)\,\sigma_n$$

where Ψ_n and σ_n are functions of x.

What is Ψ_n ? K_n?

Convergence Analysis

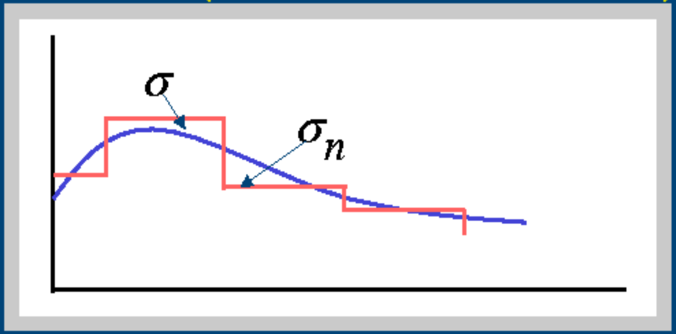
Second Kind Theory

Discrete Equivalent for Galerkin

Representation $\sigma_n(x) = \sum_{i=1}^{n} \sigma_{ni}\varphi_i(x)$

Projection $\sigma_n = P\sigma$

$$P\sigma \equiv \sum_{i=1}^{n} \overbrace{\left(\int \sigma(x)\varphi_i(x)dx \right)}^{\sigma_{ni}} \varphi_i(x)$$



Note $K\sigma_n(x) = KP\sigma(x) = \sum_{i=1}^{n} \sigma_{ni} \int G(x,x')\varphi_i(x')dx'$

Convergence Analysis

Second Kind Theory

Discrete Equivalent for Galerkin, contd..

$$P(KP\boldsymbol{\sigma}) = \sum_{j=1}^{n} \left(\int \varphi_j(x) KP\boldsymbol{\sigma}(x) dx \right) \varphi_j(x)$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \boldsymbol{\sigma}_{ni} \int \int \varphi_j(x) G(x, x') \varphi_i(x') dx dx' \right) \varphi_j(x)$$

or

$$(I + PKP)\boldsymbol{\sigma}_n = P\Psi$$

$$(I + K_n)\boldsymbol{\sigma}_n = \Psi_n$$

Given $(I + K)\sigma = \Psi$ and $||(I + K)^{-1}|| < C$

(Equation uniquely solvable)

$$(I + K_n)\sigma_n = \Psi_n$$

(Discrete Equivalent)

Consistency:

If $lim_{n\to\infty} max_{||\sigma_{smooth}||=1} ||(K - K_n)\sigma|| \to 0$

and $lim_{n\to\infty} ||\Psi - \Psi_n|| \to 0$

Then

$lim_{n\to\infty} \, ||\sigma - \sigma_n|| \to 0$

Convergence Analysis

Second Kind Theory

Rough Proof

Operator Form for the integral equation

$$\left(I + \mathrm{K}\right)\sigma = \Psi$$

Discretized Integral Equation

$$\left(I + \underbrace{\mathrm{K}_n}_{\text{discretized integral operator}}\right)\underbrace{\sigma_n}_{\text{discretized density}} = \Psi_n$$

Ignore for simplicity

Subtracting

$$\left(I + \mathrm{K}_n\right)\left(\sigma_n - \sigma\right) + \left(\mathrm{K}_n - \mathrm{K}\right)\sigma + (\Psi_n - \Psi) = 0$$

$$\Rightarrow (\sigma_n - \sigma) = (I + K_n)^{-1}\left[(K - K_n)\sigma + (\Psi - \Psi_n)\right]$$

Convergence Analysis

Second Kind Theory

Rough Proof Continued

The equation for the solution error (previous slide)

$$\underbrace{(\sigma_n - \sigma)}_{solution\ error} = (I + K_n)^{-1}(K - K_n)\sigma$$

Taking norms

$$\underbrace{||\sigma_n - \sigma||}_{\text{Error which should go to zero as n increases}} \leq \underbrace{||(I + K_n)^{-1}||}_{\text{Needs a bound, that is stability}} \underbrace{||(K - K_n)\sigma||}_{\text{Goes to zero by consistency}}$$

Convergence Analysis

Second Kind Theory

Stability Bound

Norm of solution error

$$||(\sigma_n - \sigma)|| \leq ||(I + K_n)^{-1}|| \, ||(K - K_n)\sigma||$$

Deriving the stability bound

$$(I + K_n)^{-1} = (I + K - (K - K_n))^{-1} = (I + K)^{-1}\left(I - (I + K)^{-1}(K - K_n)\right)^{-1}$$

Taking norms

$$||(I + K_n)^{-1}|| \leq \underbrace{||(I + K)^{-1}||}_{\substack{\text{Bounded by C} \\ \text{by Assumption}}} \, \left|\left| \left(I - (I + K)^{-1}(K - K_n)\right)^{-1} \right|\right|$$

Convergence Analysis

Second Kind Theory

Stability Bound Contd...

Repeating from last slide

$$\|(I+K_n)^{-1}\| \leq \underbrace{\|(I+K)^{-1}\|}_{\substack{\text{Bounded by C} \\ \text{by Assumption}}} \left\| \left(I-(I+K)^{-1}(K-K_n)\right)^{-1} \right\|$$

Bounding terms

$$\|(I+K_n)^{-1}\| \leq \frac{C}{1-\underbrace{\|(I+K)^{-1}(K-K_n)\|}_{\substack{\text{Less than } \epsilon \text{ for n larger} \\ \text{than } n_0 \text{ by consistency}}}} \leq \frac{C}{1-\epsilon} < C \text{ for } n \geq n_0$$

Convergence Analysis

Second Kind Theory

Rough Proof Completed

Final result

$$lim_{n\to\infty}||(\sigma_n - \sigma)|| \leq C\, lim_{n\to\infty}||(K - K_n)\sigma|| = 0$$

What does this mean?

The discretization convergence of a second kind integral equation solver depends on how well the integral is approximated.

Nystrom Method

1D Second Kind Example

Collocation Discretization of 1D Equation

Integral Equation

$$\Psi(x) = \sigma(x) + \int_{-1}^{1} G(x, x')\sigma(x')dS' \qquad x \in [-1, 1]$$

Apply **quadrature** to **Collocation equation**

$$\Psi(x_i) = \sigma(x_i) + \int_{-1}^{1} G(x_i, x')\sigma(x')dS'$$

$$\Rightarrow \Psi(x_i) = \sigma(x_i) + \sum_{j=1}^{n} w_j G(x_i, x_j)\sigma(x_j)$$

x_i is a collocation point

x_j's are quadrature points

Now set **quadrature points = collocation points**

Nystrom Method

1D Second Kind Example

Collocation Discretization of 1D Equation, Contd...

Set **quadrature points = collocation points**

$$\Psi(x_1) = \sigma_{n1} + \sum_{j=1}^{n} w_j G(x_1, x_j)\sigma_{nj}$$

$$\vdots$$

$$\Psi(x_n) = \sigma_{n1} + \sum_{j=1}^{n} w_j G(x_n, x_j)\sigma_{nj}$$

System of n equations in n unknowns

- Collocation equation per quad/colloc point
- Unknown density per quad/colloc point

Nystrom Method

1D Second Kind Example

1D Discretization-Matrix Comparison

Nystrom Matrix

$$\begin{bmatrix} 1+w_1 G(x_1,x_1) & \cdots & w_n G(x_1,x_n) \\ \vdots & \ddots & \vdots \\ w_1 G(x_n,x_1) & \cdots & 1+w_n G(x_n,x_n) \end{bmatrix} \begin{bmatrix} \sigma_{n1} \\ \vdots \\ \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \Psi(x_1) \\ \vdots \\ \Psi(x_n) \end{bmatrix}$$

Piecewise Constant Collocation Matrix

$$\begin{bmatrix} 1+\int_{x_0}^{x_1} G(x_{c_1},x')dS' & \cdots & \int_{x_{n-1}}^{x_n} G(x_{c_1},x')dS' \\ \vdots & \ddots & \vdots \\ \int_{x_0}^{x_1} G(x_{c_n},x')dS' & \cdots & 1+\int_{x_{n-1}}^{x_n} G(x_{c_n},x')dS' \end{bmatrix} \begin{bmatrix} \sigma_{n1} \\ \vdots \\ \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \Psi(x_{c_1}) \\ \vdots \\ \Psi(x_{c_n}) \end{bmatrix}$$

Nystrom Method

1D Second Kind Example

1D Discretization-Matrix Comparison, Contd..

Nystrom Matrix

Just Green's function evals - No integrals

Entries each have a quadrature weight

Collocation points are quadrature points

High order quadrature=faster convergence?

Piecewise Constant Collocation Matrix

Integrals of Green's function over line sections

Distant terms equal Green's function

Collocation points are basis function centroids

Low order method always

Nystrom Method

1D Second Kind Example

K_n and Ψ_n for Nystrom Method

$$K_n\sigma = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} w_j G(x_i, x_j)\sigma(x_j) \right) \varphi_i(x)$$

$$\Psi_n = \sum_{i=1}^{n} \Psi(x_i)\varphi_i(x)$$

Convergence Theorem

In the limit as $n \rightarrow \infty$ (number of quad points $\rightarrow \infty$)
The discretization error =
$$max_{||\sigma||=1} ||(K - K_n)\sigma|| \rightarrow 0$$

AT THE SAME RATE as the underlying quadrature!!

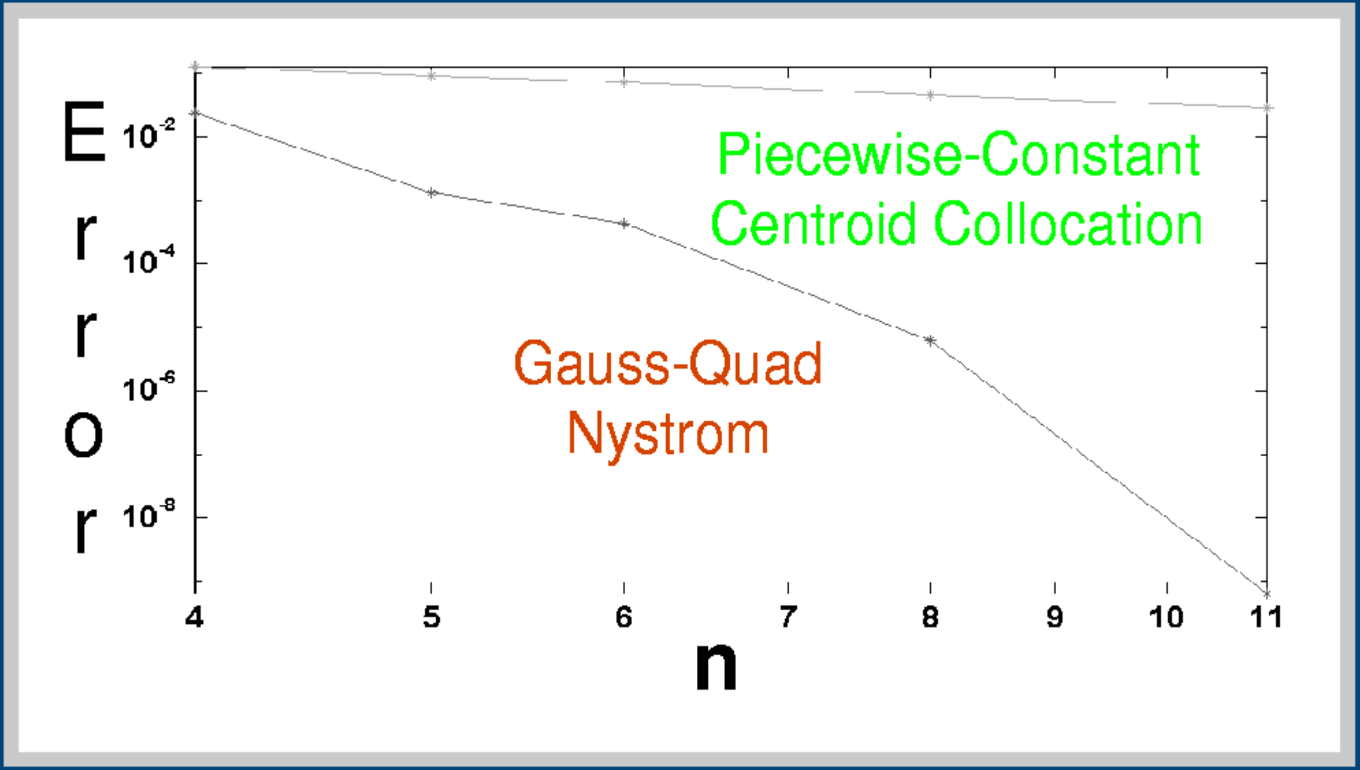
Gauss Quadrature $\Rightarrow$ Exponential Convergence!

Nystrom Method

1D Second Kind Example

Convergence Comparison

$$\cos 2\pi x = \sigma(x) + \int_{-1}^{1}(x - x')^2\sigma(x')dS'$$



Nystrom Method

1D Second Kind Example

Convergence Caveat

If Nystrom method can have exponential convergence, why use anything else?

Gaussian quadrature has exponential convergence for **nonsingular** kernels

Most physical problems of interest have **singular kernels** ($1/r$, $\exp ikr/r$, etc)

Summary

Integral Equation Methods

Reviewed Galerkin and Collocation

Example of Convergence Issues in 1D

1st and 2nd kind integral equations, null spaces

Convergence for second kind equations

Show consistency and stability issues

Nystrom methods

High order convergence

Did not address singular integrands