Discretization of the Poisson Problem in $\mathbb{R}^1$: Formulation

April 2, 2003

Model Problems

Dirichlet

Strong Form

Domain: $\Omega = (0, 1)$.

Find u such that

$$-u_{xx} = f \quad \text{in } \Omega$$

$$u(0) = u(1) = 0 \quad ,$$

for given f.

Model Problems

Dirichlet

Minimization Statement

Define $X \equiv H_0^1(\Omega)$.

Find

$$\boxed{u = \arg\min_{w \in X} J(w)}$$

where

$$J(w) = \frac{1}{2} \int_0^1 w_x^2 \, dx - \int_0^1 f w \, dx \; .$$

Model Problems

Dirichlet

Weak Formulation

Find $u \in X$ such that

$$\delta J_v(u) = 0 \,, \qquad \forall\, v \in X$$

$$\Updownarrow$$

$$\boxed{\int_0^1 u_x\, v_x\, dx = \int_0^1 f\, v\, dx \,, \qquad \forall\, v \in X \,.}$$

Model Problems

Dirichlet

Notation

Define

$$a(w,v) = \int_0^1 w_x\, v_x\, dx$$

$$\ell(v) = \int_0^1 f\, v\, dx\ .$$

Minimization:

$$u = \arg \min_{w \in X} \frac{1}{2} a(w,w) - \ell(w)$$

Weak:

$$u \in X\colon\ a(u,v) = \ell(v)\ ,\ \forall\, v \in X$$

Model Problems

Dirichlet

Generalization

For any $\ell(v) \in H^{-1}(\Omega)$,

find $u \in H_0^1(\Omega)$ such that

$$u = \arg \min_{w \in H_0^1(\Omega)} \frac{1}{2} a(w, w) - \ell(w) \; ; \quad \text{or}$$

$$a(u, v) = \ell(v), \qquad \forall \, v \in H_0^1(\Omega) \; ;$$

for example, $\ell(v) = \langle \delta_{x_0}, v \rangle = v(x_0)$ is admissible.

Model Problems

Dirichlet

Regularity

If $\ell \in H^{-1}(\Omega)$,

$$\|u\|_{H^1(\Omega)} \leq C \, \|\ell\|_{H^{-1}(\Omega)} \, .$$

If $\ell \in L^2(\Omega)$, $\ell(v) = \int_0^1 f \, v \, dx$

$$\|u\|_{H^2(\Omega)} \leq C_0 \, \|f\|_{L^2(\Omega)} \, .$$

N1

Model Problems

"Neumann"

Strong Form

Domain: $\Omega = (0, 1)$.

Find u such that

$$-u_{xx} = f \qquad \text{in } \Omega ,$$

$$u(0) = 0 ,$$

$$u_x(1) = g ,$$

for given f, g .

Model Problems

"Neumann"

Minimization Statement

Define $X \equiv \left\{ v \in H^1(\Omega) \mid v(0) = 0 \right\}$.

Find

$$\boxed{u = \arg \min_{w \in X} J(w)}$$

where

$$J(w) = \frac{1}{2} \int_0^1 w_x^2 \, dx - \int_0^1 f \, w \, dx - g \, w(1) \, .$$

Model Problems

"Neumann"

Weak Statement

Find $\boldsymbol{u} \in \boldsymbol{X}$ such that

$$\delta J_v(u) = 0 \,, \qquad \forall\, v \in X$$

$$\Updownarrow$$

$$\int_0^1 u_x\, v_x\, dx = \int_0^1 f\, v\, dx + g\, v(1) \,, \qquad \forall\, v \in X \,.$$

Model Problems

"Neumann"

Notation

Define
$$a(w,v) = \int_0^1 w_x \, v_x \, dx$$

$$\ell(v) = \int_0^1 f \, v \, dx + g \, v(1) \, .$$

N2

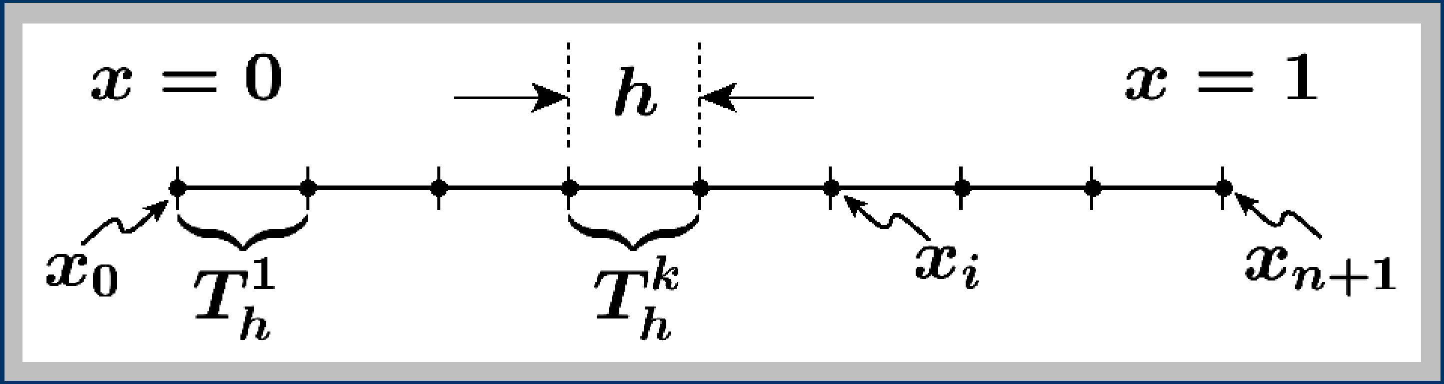
Minimization:
$$u = \arg \min_{w \in X} \frac{1}{2} a(w,w) - \ell(w)$$

Weak:
$$u \in X: \; a(u,v) = \ell(v) \, , \; \forall \, v \in X$$

Rayleigh-Ritz Approach

Approximation

Mesh



$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{T}_h^k \qquad T_h^k \,,\, k = 1, \ldots, K = n+1: \ \textit{elements}$$

$$x_i \,,\, i = 0, \ldots, n+1: \ \textit{nodes}$$

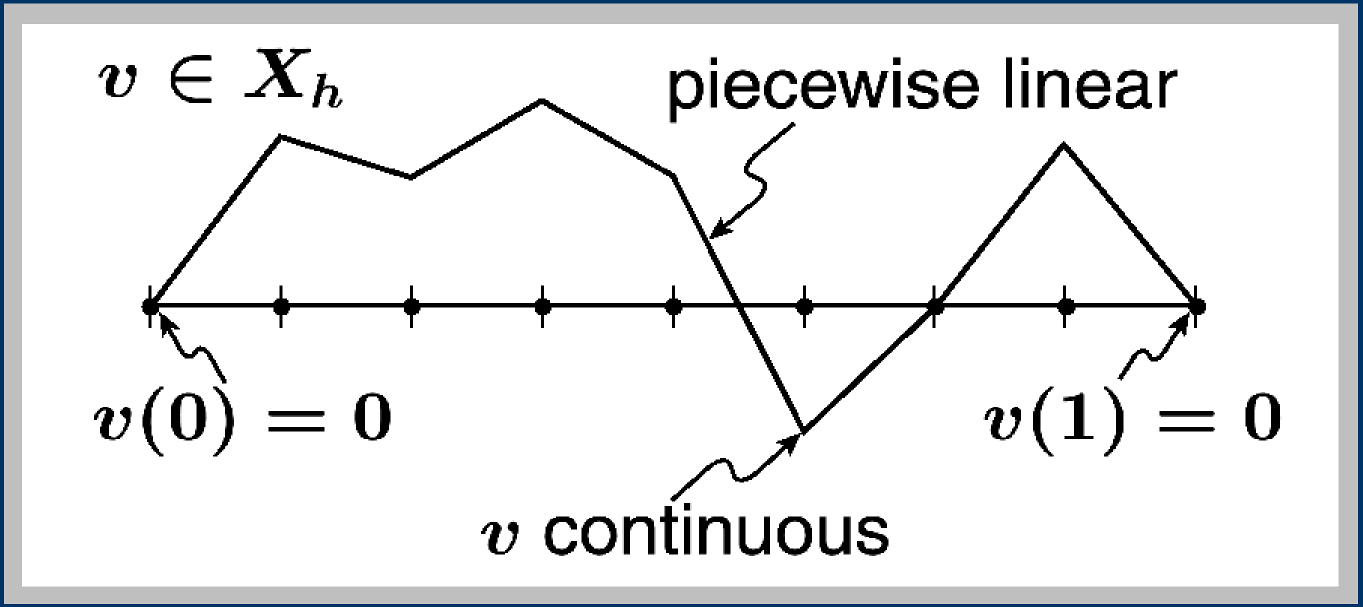
N3

Rayleigh-Ritz Approach

Approximation

Space $X_h \subset X$

$$X_h = \left\{ v \in X \;\middle|\; v|_{T_h^k} \in \mathbb{P}_1(T_h^k), \quad k = 1, \dots, K \right\}$$



N4

Rayleigh-Ritz Approach

Approximation

Basis...

General definition: given a linear space Y,

a set of members $y_j \in Y$, $j = 1, \ldots, M$,

is *a* basis for Y if and only if

$\forall\, y \in Y$, $\exists$ unique $\alpha_j \in \mathbb{R}$ such that

$$y = \sum_{j=1}^{M} \alpha_j\, y_j \;;$$

$\mathbf{dim}$(ension) $(Y) = M$.

N5 N6 E1 E2

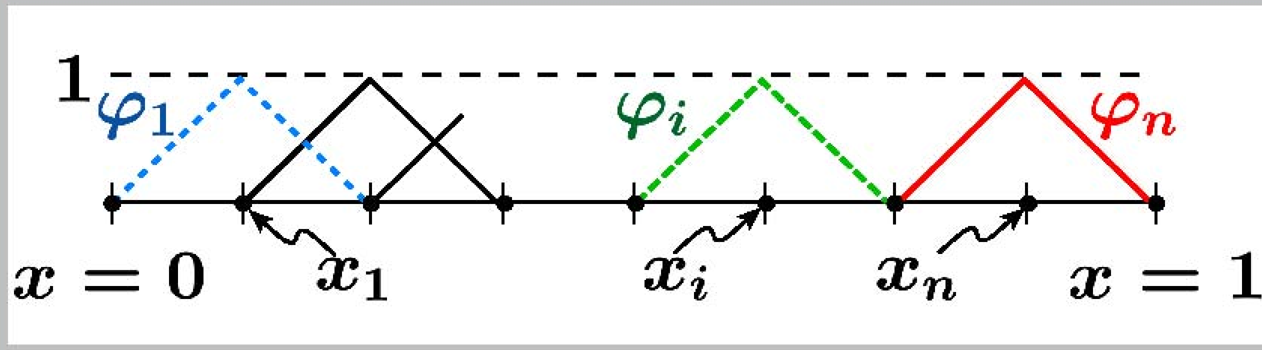
Rayleigh-Ritz Approach

Approximation

...Basis

Nodal basis for X_h:

$$\varphi_j,\ j = 1, \ldots, n = \dim(X_h)$$



φ_i nonzero only on $\overline{T_h^i} \bigcup \overline{T_h^{i+1}}$

N7 N8

Rayleigh-Ritz Approach

"Projection"

Plan...

Let

$$\underbrace{u_h \quad (\in X_h)}_{\text{RR/FE Approximation}} = \sum_{j=1}^{n} u_{hj}\, \varphi_j(x) \; ;$$

set $u_{hj} = w_j$ that minimize

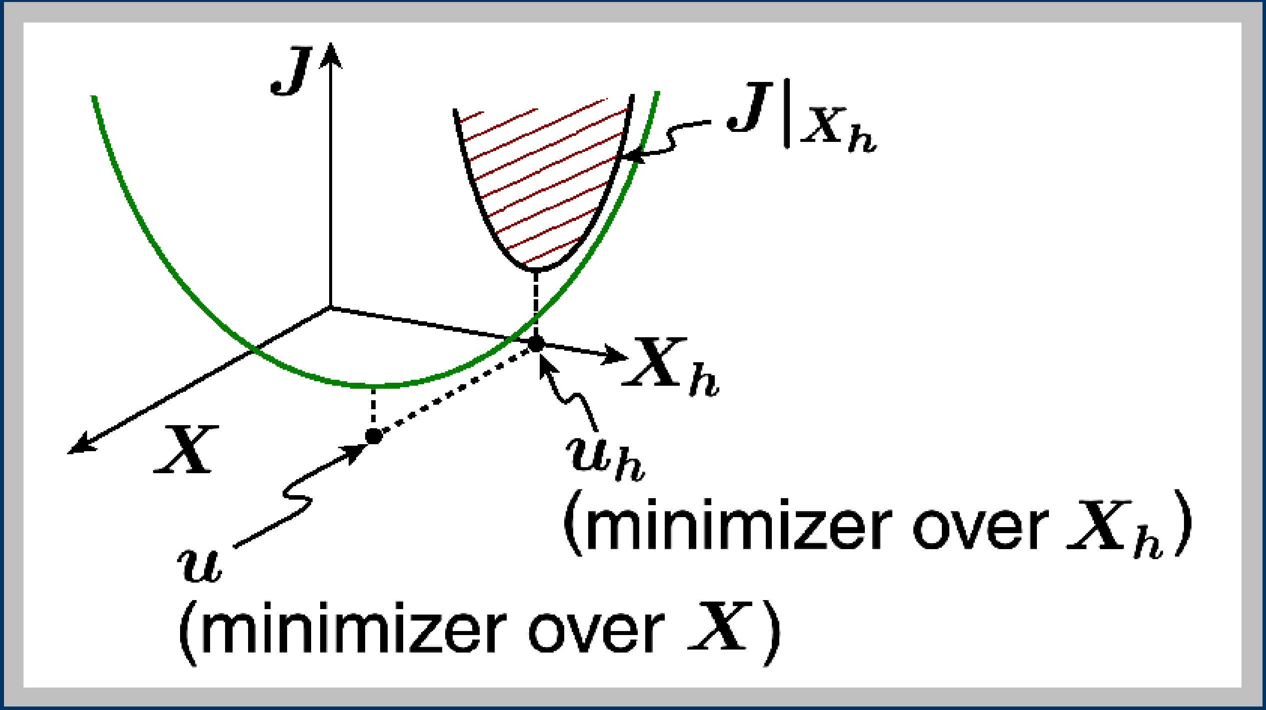
$$J\left(\sum_{j=1}^{n} w_j\, \varphi_j\right) .$$

Rayleigh-Ritz Approach

"Projection"

...Plan

Geometric Picture:



Rayleigh-Ritz Approach

"Projection"

$J|_{X_h}$...

$$J\left(\sum_{j=1}^{n} w_j\,\varphi_j\right) = \frac{1}{2}\,a\left(\sum_{i=1}^{n} w_i\,\varphi_i, \sum_{j=1}^{n} w_j\,\varphi_j\right) - \ell\left(\sum_{i=1}^{n} w_i\,\varphi_i\right)$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i\,a(\varphi_i,\varphi_j)\,w_j - \sum_{i=1}^{n} w_i\,\ell(\varphi_i)$$

by *bilinearity* and *linearity*.

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Rayleigh-Ritz Approach

"Projection"

...$J|_{X_h}$

$$J^R(\underline{w} \in \mathbb{R}^n) \equiv J\left(\sum_{j=1}^{n} w_j\, \varphi_j\right)$$

$$= \frac{1}{2}\,\underline{w}^T\,\underline{A}_h\,\underline{w} - \underline{w}^T\,\underline{F}_h\,.$$

$$\underline{F}_h \in \mathbb{R}^n\!: \; F_{h\,i} \equiv \ell(\varphi_i) \left(= \int_\Omega f\,\varphi_i\,dx\right)$$

$$\underline{A}_h \in \mathbb{R}^{n\times n}\!: \; A_{h\,ij} \equiv a(\varphi_i, \varphi_j) = \int_\Omega \frac{d\varphi_i}{dx}\,\frac{d\varphi_j}{dx}\,dx$$

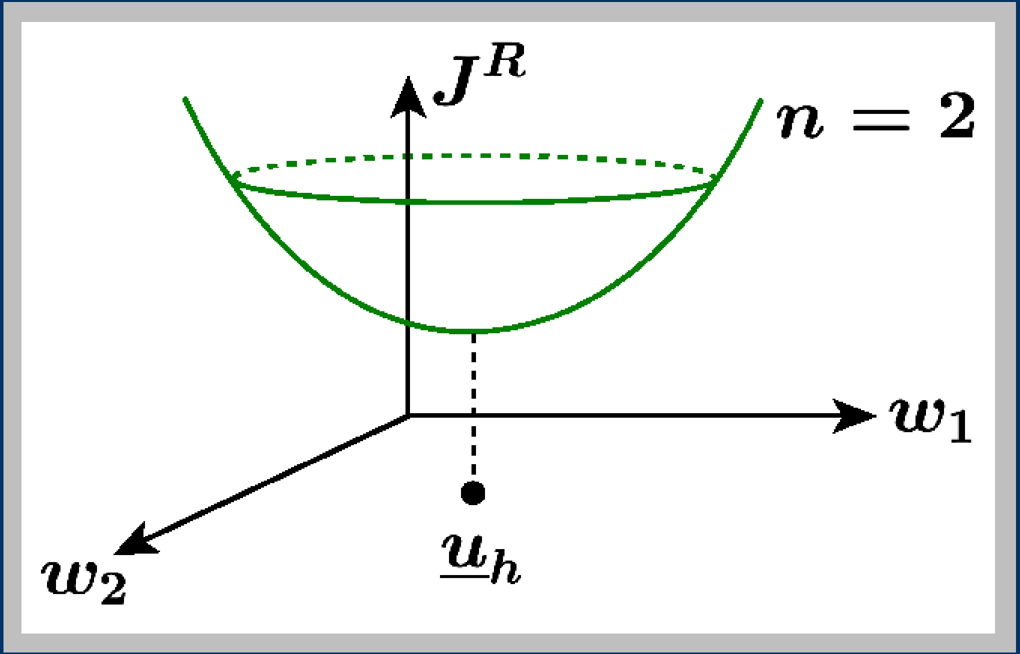
E3

Rayleigh-Ritz Approach

"Projection"

Minimization...

$$\underline{u}_h = \arg \min_{\underline{w} \in \mathbb{R}^n} J^R(\underline{w})$$



Expand $J^R(\underline{w} = \underline{u}_h + \underline{v})$; require $J^R(\underline{w}) > J^R(\underline{u}_h)$ unless $\underline{v} = \mathbf{0}$.

Rayleigh-Ritz Approach

"Projection"

...Minimization...

$$J^R(\underline{u}_h + \underline{v})$$

$$= \frac{1}{2}(\underline{u}_h + \underline{v})^T \underline{A}_h(\underline{u}_h + \underline{v}) - (\underline{u}_h + \underline{v})^T \underline{F}_h$$

$$= \frac{1}{2}\underline{u}_h^T \underline{A}_h \underline{u}_h - \underline{u}_h^T \underline{F}_h$$

$$+ \frac{1}{2}\underline{v}^T \underline{A}_h \underline{u}_h + \frac{1}{2}\underline{u}_h^T \underline{A}_h \underline{v} - \underline{v}^T \underline{F}_h$$

$$+ \frac{1}{2}\underline{v}^T \underline{A}_h \underline{v}$$

Rayleigh-Ritz Approach

"Projection"

...Minimization...

$$J^R(\underline{u}_h + \underline{v}) = J^R(\underline{u})$$

$$+\underbrace{(\underline{A}_h\,\underline{u}_h - \underline{F}_h)^T}_{\nabla J^R(\underline{u}_h)}\,\underline{v} \quad \delta J^R_{\underline{v}}(\underline{u}_h)$$

SPD

$$+\frac{1}{2}\,\underbrace{\underline{v}^T\,\underline{A}_h\,\underline{v}}_{>0,\ \forall\,\underline{v}\neq 0}$$

SPD

Rayleigh-Ritz Approach

"Projection"

...Minimization

If (and only if)

$$\delta J^R_{\underline{v}}(\underline{u}_h) = 0, \quad \forall\, \underline{v} \in \mathbb{R}^n$$

$$\Updownarrow$$

$$\nabla J^R(\underline{u}_h) = \underline{A}_h\, \underline{u}_h - \underline{F}_h = \underline{0}$$

then

$$J^R(\underline{w} = \underline{u}_h + \underline{v}) > J^R(\underline{u}_h)\ , \qquad \forall\, \underline{v} \neq 0\ .$$

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Rayleigh-Ritz Approach

"Projection"

Final Result

Find $\underline{u}_h \in \mathbb{R}^n$ such that

$$\underbrace{\underline{A}_h}_{a(\varphi_i,\varphi_j)} \; \underline{u}_h = \underbrace{\underline{F}_h}_{\ell(\varphi_i)} \quad \Rightarrow \quad u_h(x) = \sum_{j=1}^{N} u_{hj} \, \varphi_j(x) \; .$$

SPD $\Rightarrow$ existence and uniqueness.

Galerkin Approach

Approximation

Triangulation $\mathcal{T}_h$;

Space X_h ; and

(Nodal) Basis $X_h = \mathbf{span}\,\{\varphi_1, \ldots, \varphi_n\}$;

as for the Rayleigh-Ritz approach.

Galerkin Approach

Projection

Plan...

Let

$$u_h(\in X_h) = \sum_{j=1}^{n} u_{hj}\, \varphi_j(x) \; ;$$

set u_{hj} such that

$$\delta J_v(u_h) = 0 \; , \qquad \forall\, v \in X_h$$

$$\Updownarrow$$

$$a(u_h, v) = \ell(v) \; , \qquad \forall\, v \in X_h \; .$$

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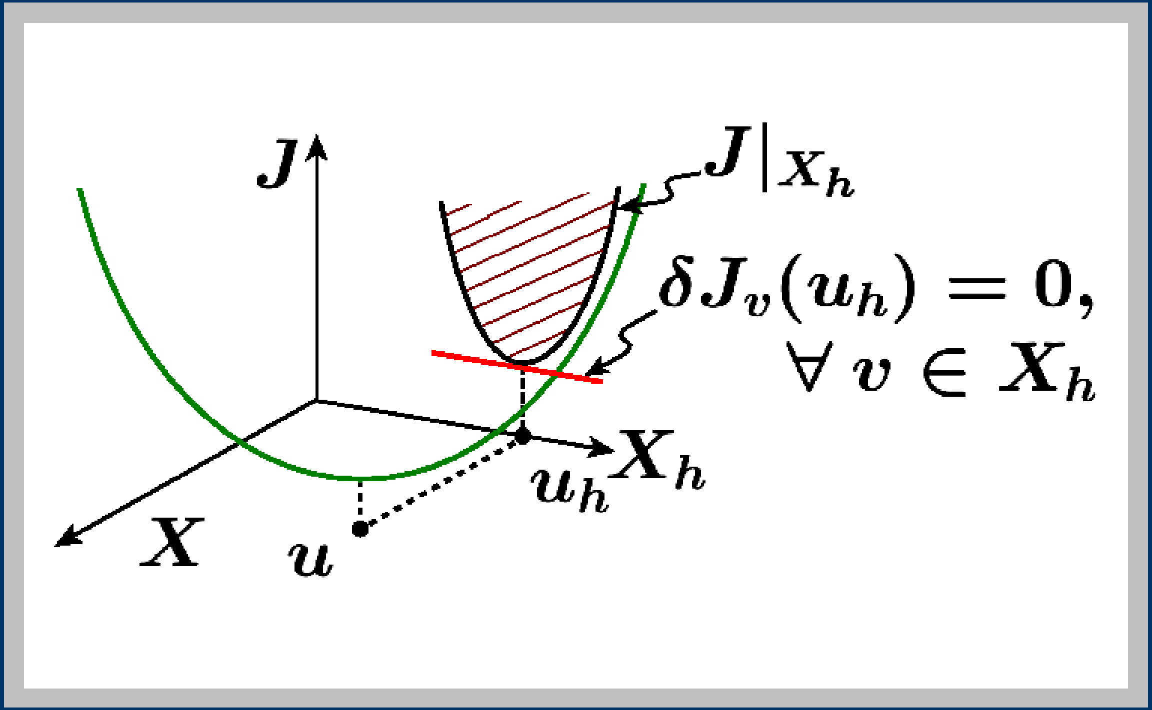
Galerkin Approach

Projection

...Plan

Geometric Picture:



Galerkin Approach

Projection

Variation...

Since *any* $v \in X_h$ can be written as

$$v = \sum_{i=1}^{n} v_i \, \varphi_i(x) \; ,$$

$$a(u_h, v) = \ell(v) \; , \qquad \forall \, v \in X_h$$

$$\Updownarrow$$

$$a\left(u_h, \sum_{i=1}^{n} v_i \, \varphi_i\right) = \ell\left(\sum_{i=1}^{n} v_i \, \varphi_i\right) \; , \qquad \forall \, \underline{v} \in \mathbb{R}^n \; .$$

Galerkin Approach

Projection

...Variation...

But $u_h = \sum_{i=1}^{n} u_{hj}\,\varphi_j$, so

$$a\left(\sum_{j=1}^{n} u_{hj}\,\varphi_j, \sum_{i=1}^{n} v_i\,\varphi_i\right) = \ell\left(\sum_{i=1}^{n} v_i\,\varphi_i\right), \quad \forall \underline{v} \in \mathbb{R}^n$$

or, by bilinearity and linearity

$$\underline{v}^T\,\underline{A}_h\,\underline{u}_h = \underline{v}^T\,\underline{F}_h\,, \quad \forall\,\underline{v} \in \mathbb{R}^n\,.$$

Galerkin Approach

Projection

...Variation

Take $\underline{v} = (1 \;\; 0 \;\; \ldots \;\; 0)^T \Rightarrow \sum_{j=1}^{n} A_{h\,1\,j}\, u_{h\,j} = F_{h\,1}$

$\underline{v} = (0 \;\; 1 \;\; \ldots \;\; 0)^T \Rightarrow \sum_{j=1}^{n} A_{h\,2\,j}\, u_{h\,j} = F_{h\,2}$

$$\vdots$$

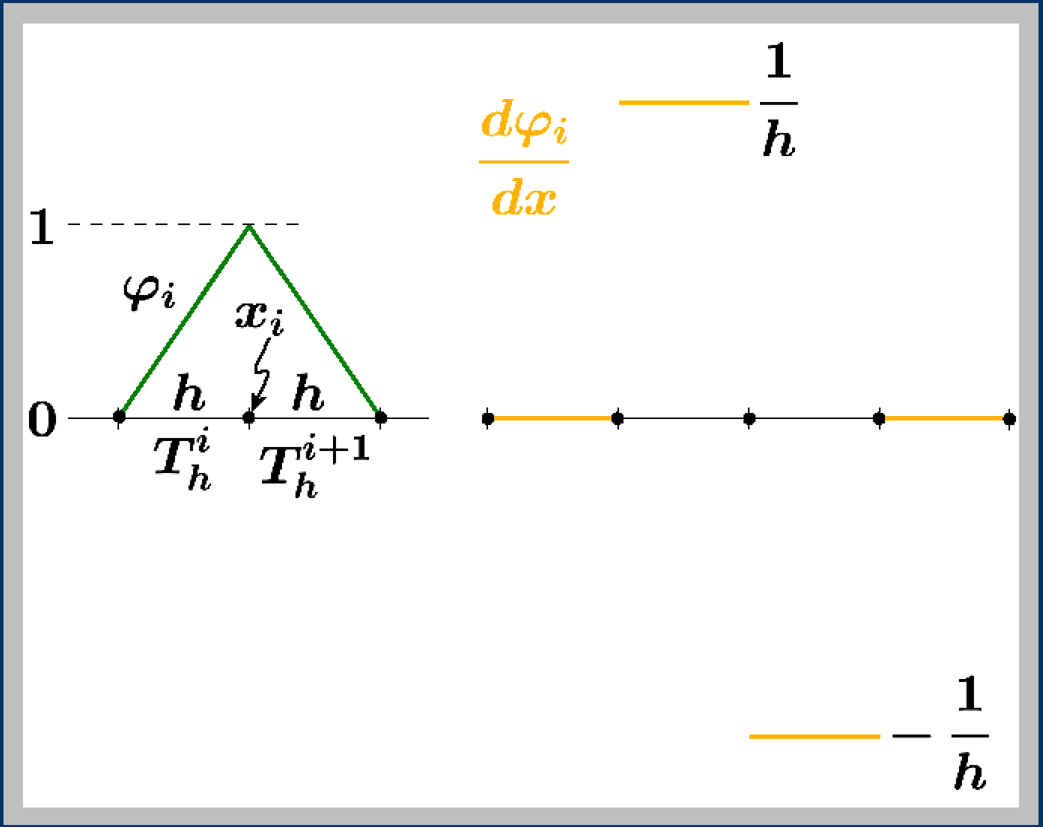
$$\underline{v}^T\, \underline{A}_h\, \underline{u}_h = \underline{v}^T\, \underline{F}_h\,,\ \forall\, \underline{v} \in \mathbb{R}^n \;\Leftrightarrow\; \underline{A}_h\, \underline{u}_h = \underline{F}_h$$

N12

Discrete Equations

Matrix Elements: $\underline{A}_h$

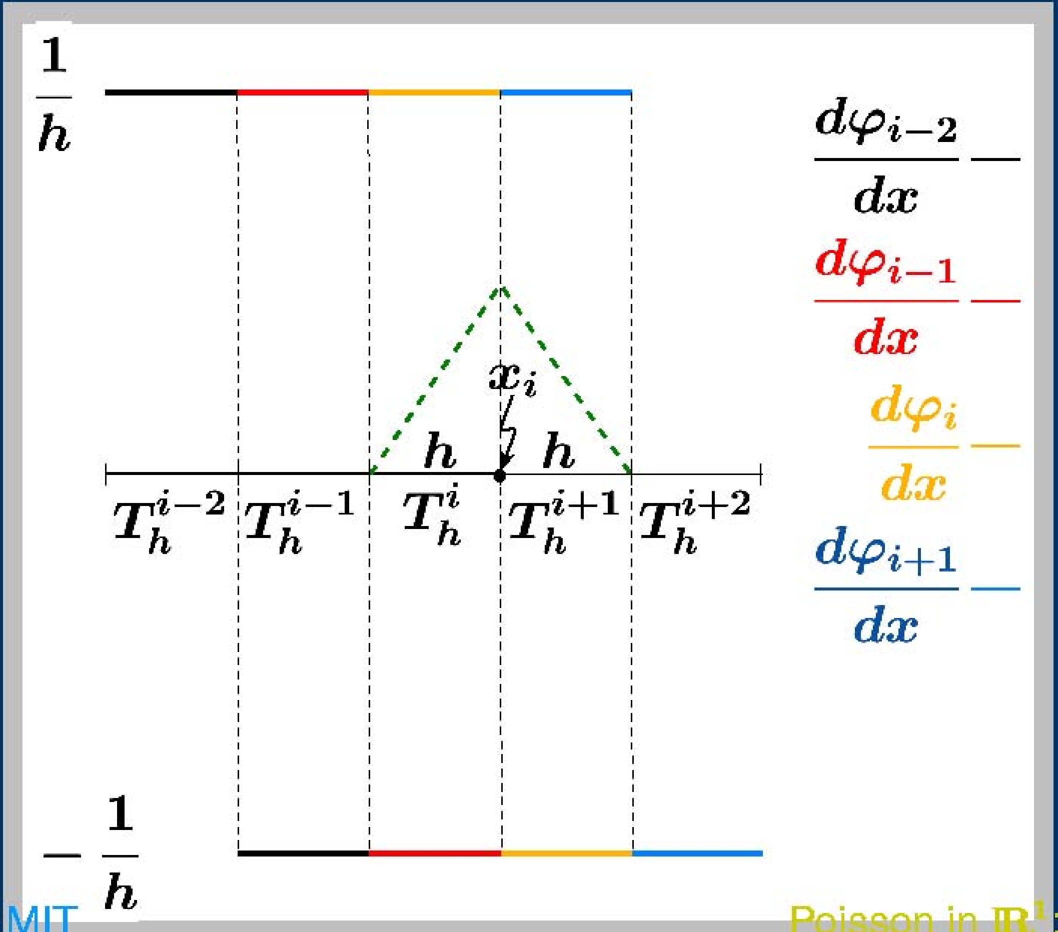
φ_i and $d\varphi_i/dx$...



Discrete Equations

Matrix Elements: $\underline{A}_h$

...φ_i and $d\varphi_i/dx$



Discrete Equations

Matrix Elements: $\underline{A}_h$

Typical Row

$$A_{h\,i\,j} = \int_\Omega \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}dx = \int_{T_h^i} \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}dx + \int_{T_h^{i+1}} \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}dx$$

is nonzero only for $j = i-1,\ i,\ i+1$

$$A_{h\,i\,i} = \tfrac{1}{h^2}\,(h) + \tfrac{1}{h^2}\,(h) = \tfrac{2}{h}$$

$$A_{h\,i\,i-1} = \tfrac{1}{h}\left(-\tfrac{1}{h}\right)(h) = -\tfrac{1}{h}$$

$$A_{h\,i\,i+1} = \left(-\tfrac{1}{h}\right)\tfrac{1}{h}\,(h) = -\tfrac{1}{h}$$

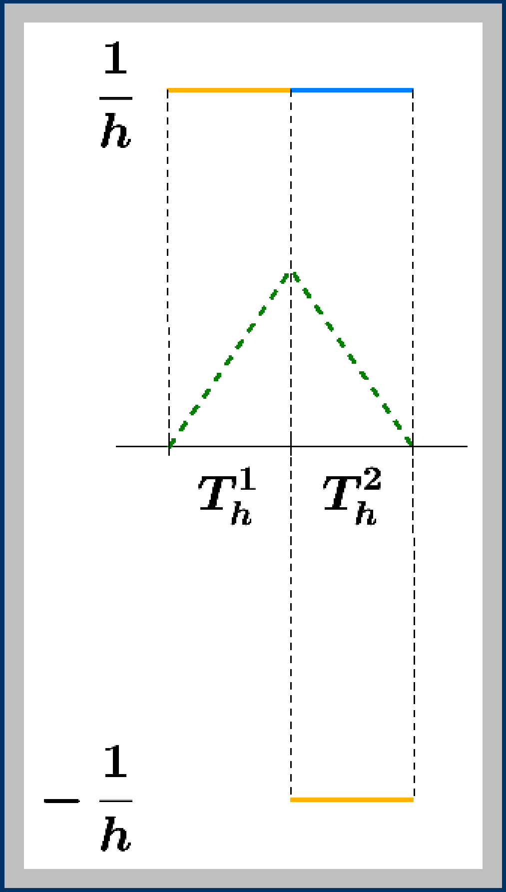
Discrete Equations

Matrix Elements: $\underline{A}_h$

Boundary Rows

$$\underline{A}_{h\,1\,1} = \frac{2}{h},\; \underline{A}_{h\,1\,2} = -\frac{1}{h}\,,$$

$$\underline{A}_{h\,n\,n} = \frac{2}{h},\; \underline{A}_{h\,n\,n-1} = -\frac{1}{h}\,.$$



Properties of $\underline{A}_h$

$$\underline{A}_h = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ 0 & & & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

$\underline{A}_h$ is SPD; *and*
diagonally dominant; *and*
sparse; *and*
tridiagonal.

Discrete Equations

“Load” Vector Elements: $\underline{F}_h$

General case, $\ell(v)$: $F_{h\,i} = \ell(\varphi_i)$

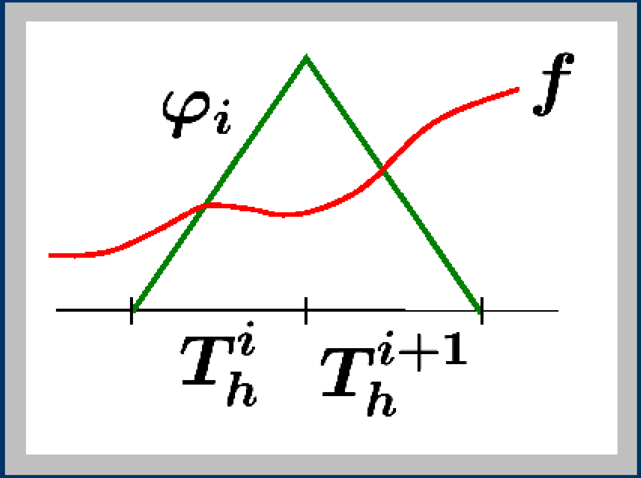
Example: $\ell(v) = \langle \delta_{x_{i^*}}, v \rangle$

$$\begin{aligned} \ell(\varphi_i) &= \langle \delta_{x_{i^*}}, \varphi_i \rangle \\ &= \varphi_i(x_{i^*}) \\ &= \delta_{i\,i^*} \,. \end{aligned}$$

“Load” Vector Elements: $\underline{F}_h$

Particular case, $\ell(v) = \int_\Omega f\, v\, dx$:

$$F_{h\,i} = \int_{T_h^i} f\, \varphi_i\, dx + \int_{T_h^{i+1}} f\, \varphi_i\, dx, \quad i = 1, \ldots, n\,;$$



Numerical quadrature — “variational crime” — next lecture.

N13

Discrete Equations

Summary

$\underline{u}_h \in \mathbb{R}^n$ satisfies

$$\frac{1}{h}\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & 0 & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}\begin{pmatrix} u_{h\,1} \\ \vdots \\ u_{h\,n} \end{pmatrix} = \begin{pmatrix} F_{h\,i} \\ \vdots \\ F_{h\,n} \end{pmatrix}$$

E4 E5 E6 E7

The Mass Matrix

Motivation

Definition

$\underline{M}_h \in \mathbb{R}^{n\times n}$:

$$M_{h\,i\,j} = \underbrace{\int_\Omega \varphi_i \varphi_j \, dx}_{\text{originating form: } (w,v)_{L^2(\Omega)}} \quad ;$$

the finite element "identity" (I) operator.

The Mass Matrix

Motivation

"Applications"

$\underline{M}_h$ appears where the identity appears

- as part of differential operator, $-u_{xx} + Iu = f$; E8
- in eigenvalue problems, $-u_{xx} = \lambda\, Iu$;
- in parabolic PDEs, $I\,\dfrac{\partial u}{\partial t} = \nabla^2 u$;
- in quadrature by interpolation.

The Mass Matrix

$\underline{M}_h$ is SPD:

$$\underline{v}^T \underline{M}\, \underline{v} = \sum_{i=1}^{n} v_i \sum_{j=1}^{n} v_j \int_0^1 \varphi_i\, \varphi_j\, dx$$

$$= \int_0^1 \sum_{i=1}^{n} v_i\, \varphi_i \sum_{j=1}^{n} v_j\, \varphi_j\, dx$$

$$= \int_0^1 \underbrace{\left(\sum_{i=1}^{n} v_i\, \varphi_i \right)^2}_{v \in X_h} dx > 0 \quad \underbrace{\text{if } \underline{v} \neq 0}_{\varphi_i \text{ are } \textit{basis}} \quad .$$

The Mass Matrix

Properties

Particular

For linear elements, nodal basis: E9

$$\underline{M}_h = h \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & & & \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \text{0} & \\ & & \ddots & & \\ & \text{0} & & \frac{2}{3} & \frac{1}{6} \\ & & & \frac{1}{6} & \frac{2}{3} \end{pmatrix}$$

sparse, banded, tri-diagonal — “close” to $\underline{I}$.