Truncation Error in Finite Difference

Backward difference in space

$$\left.\frac{\partial U}{\partial x}\right|_i^n \simeq \frac{U_i^n - U_{i-1}^n}{\Delta x}$$

$$\tau_i^n = \left.\frac{\partial U}{\partial x}\right|_i^n - \frac{\cancel{U_i^n} - U_{i-1}^n}{\Delta x}$$

$$U_{i-1}^n = \cancel{U_i^n} - \Delta x \left.\frac{\partial U}{\partial x}\right|_i^n + \frac{\Delta x^2}{2} \left.\frac{\partial^2 U}{\partial x^2}\right|_i^n + O(\Delta x^3)$$

$$\tau_i^n = \frac{\Delta x}{2} \left.\frac{\partial^2 U}{\partial x^2}\right|_i^n + O(\Delta x^2) = \underline{O(\Delta x)}$$

First Order

Truncation Error in Finite Difference

Central difference in space

$$\left.\frac{\partial U}{\partial x}\right|_i^n \simeq \frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}$$

$$\tau_i^n = \left.\frac{\partial U}{\partial x}\right|_i^n - \frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}$$

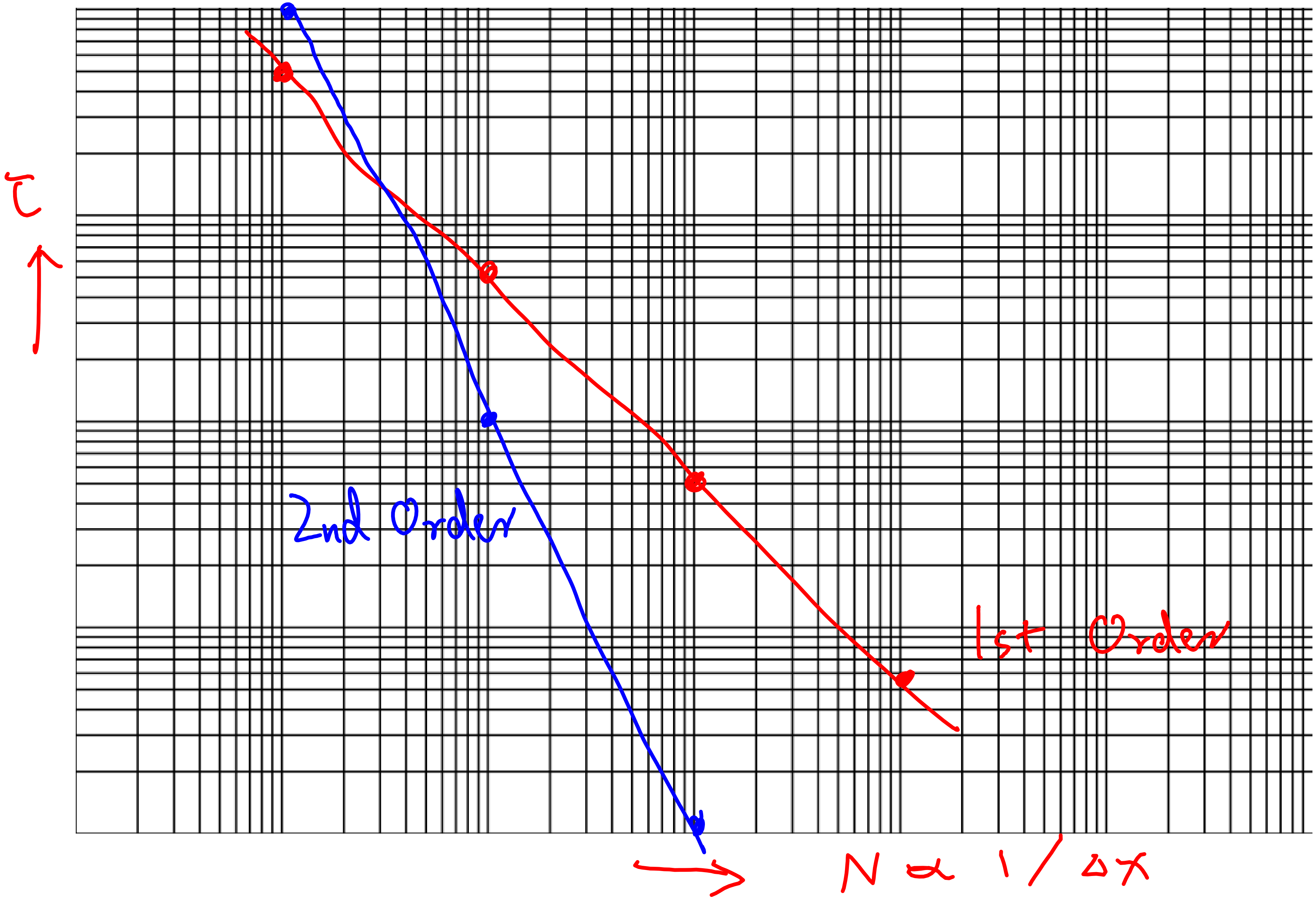
$$U_{i+1}^n = \cancel{U_i^n} + \Delta x \left.\frac{\partial U}{\partial x}\right|_i^n + \cancel{\frac{\Delta x^2}{2}\left.\frac{\partial^2 U}{\partial x^2}\right|_i^n} + \frac{\Delta x^3}{6}\left.\frac{\partial^3 U}{\partial x^3}\right|_i^n + O(\Delta x^4)$$

$$U_{i-1}^n = \cancel{U_i^n} - \Delta x \left.\frac{\partial U}{\partial x}\right|_i^n + \cancel{\frac{\Delta x^2}{2}\left.\frac{\partial^2 U}{\partial x^2}\right|_i^n} - \frac{\Delta x^3}{6}\left.\frac{\partial^3 U}{\partial x^3}\right|_i^n + O(\Delta x^4)$$

$$\tau_i^n = -\frac{\Delta x^2}{6}\left.\frac{\partial^3 U}{\partial x^3}\right|_i^n + O(\Delta x^3) = \underline{O(\Delta x^2)}$$

Second Order

Truncation Error in Finite Difference



Error in Finite Difference Solution (Global Error)

$U(x,t)$: exact solution

U_i^n : finite difference solution

$$e_i^n = U_i^n - U(x_i, t_n)$$

$$\|e\|_\infty := \max_{\substack{\text{All } i \\ \text{All } t_n < T}} |e_i^n| \quad (L\text{-}\infty \text{ norm})$$

A global measure of solution error

$$\|e\| = O(\Delta x^p) + O(\Delta t^q)$$

order of spatial discretization   order of time discretization

Consistency, Stability, Convergence

Consistency := $\tau_i^n \xrightarrow{\Delta x \to 0 \,\&\, \Delta t \to 0} 0$

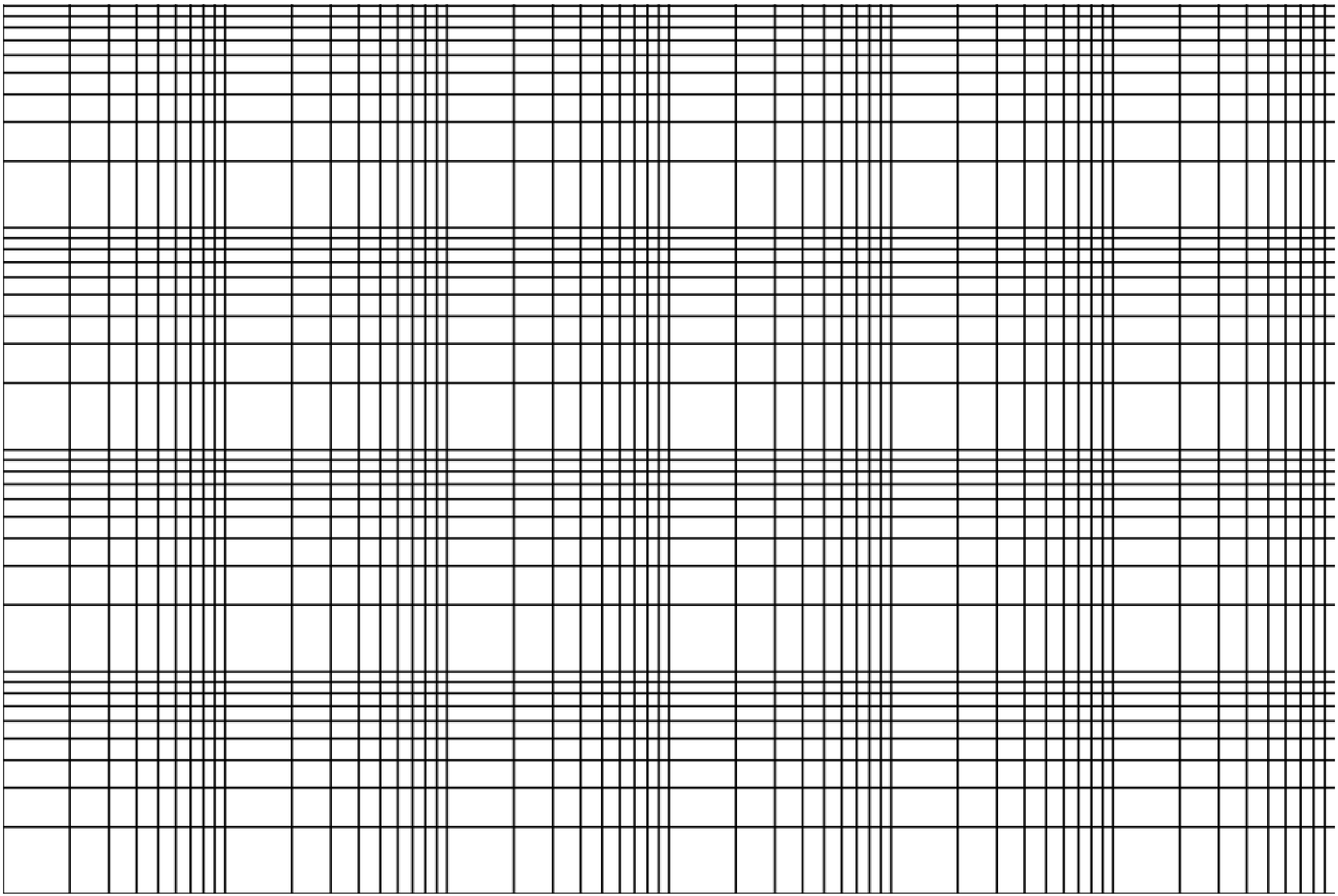
Stability := solution won't diverge as $\Delta x \to 0$ & $\Delta t \to 0$

(More on this soon)

$\|e\| \xrightarrow{\Delta x \to 0 \,\&\, \Delta t \to 0} 0$

$\star$

Convergence of Finite Difference Solution



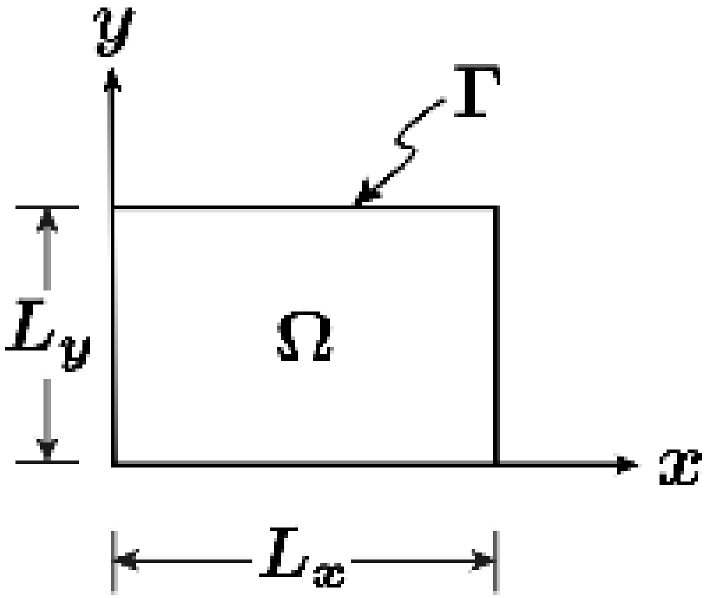
Finite Difference for Multi-D Partial Differential Equations

Example: Advection in 2D:

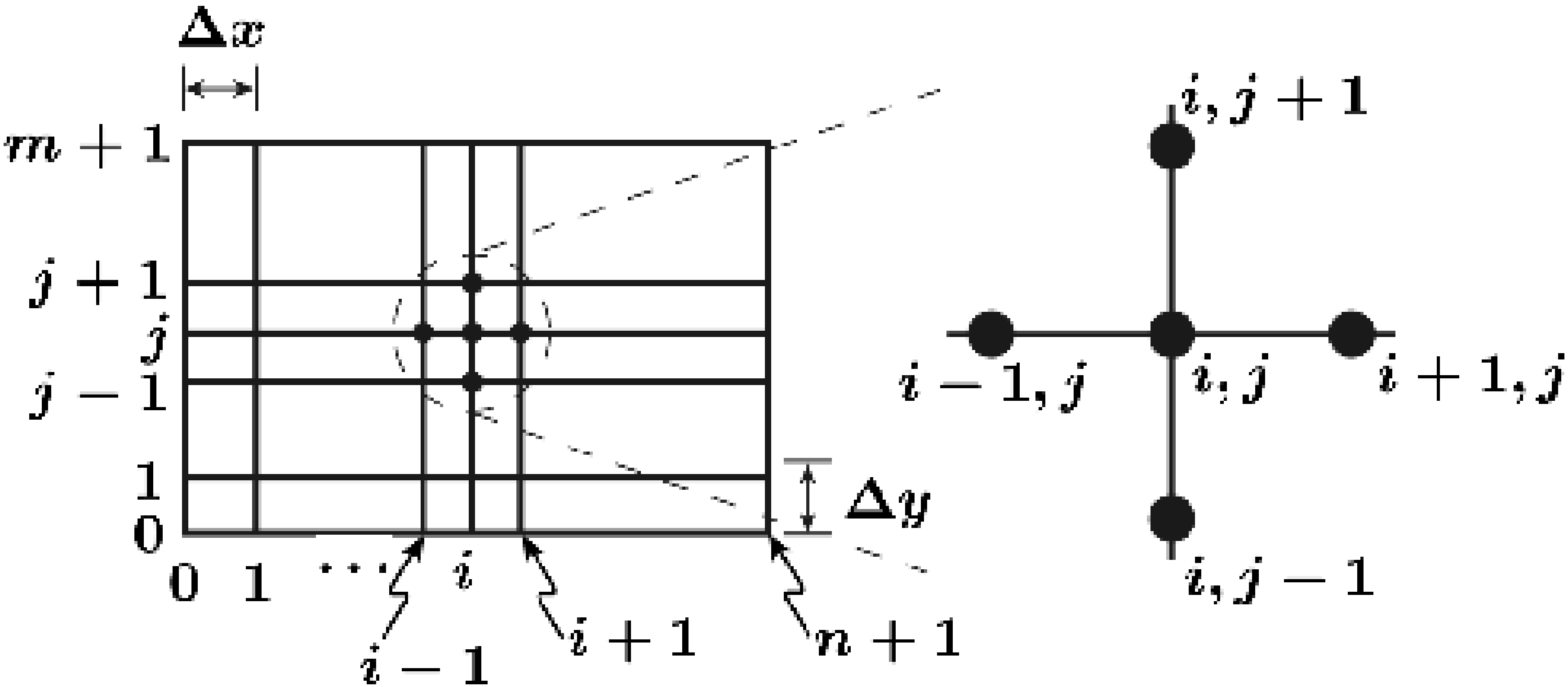
$$\frac{\partial U}{\partial t} + c_x \frac{\partial U}{\partial x} + c_y \frac{\partial U}{\partial y} = 0$$

Conservation Law:

$$\vec{F} = (c_x U, \ c_y U)$$

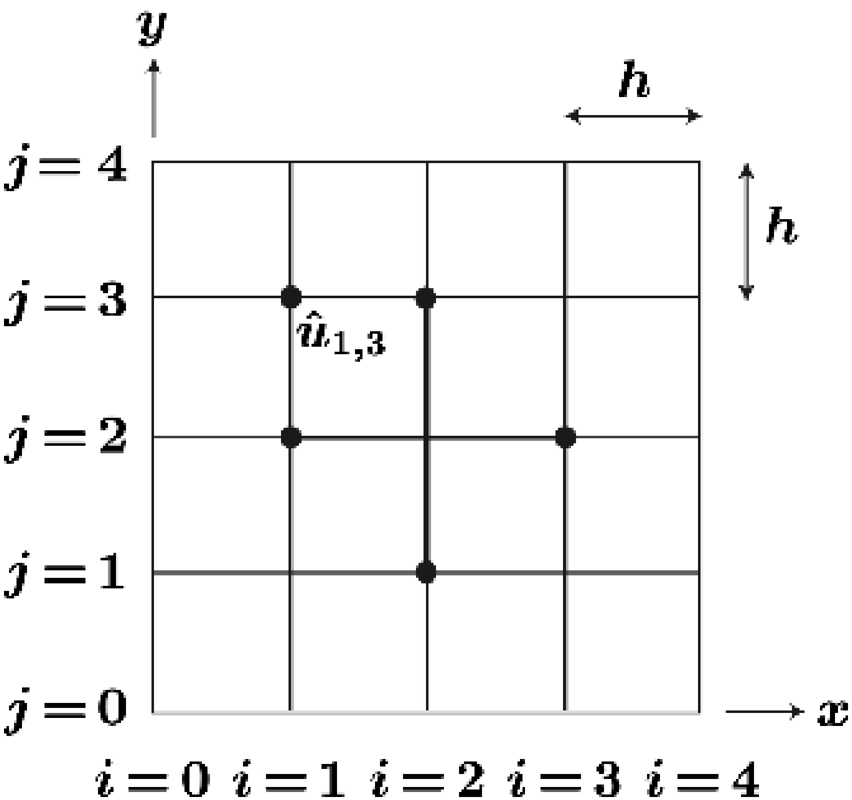


FD Discretization of in 2D



$$U_{i,j}^{n} \approx U(x_i, y_j, t_n)$$

FD for 2D Advection Equation



$$\left.\frac{\partial U}{\partial x}\right|_{i,j} \approx \frac{U_{i,j} - U_{i-1,j}}{\Delta x}$$

$$\left.\frac{\partial U}{\partial y}\right|_{i,j} \approx \frac{U_{i,j} - U_{i,j-1}}{\Delta y}$$

Backward-in-space

$$\left.\frac{\partial U}{\partial x}\right|_{i,j} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central-in-space

$$\left.\frac{\partial U}{\partial y}\right|_{i,j} \approx \frac{U_{i,j+1} - U_{i,j-1}}{2\Delta y}$$

Matrix form

$\frac{\partial U}{\partial x}$: Back-in-space

$$
\left(\begin{array}{ccc|ccc|c|cc}
\frac{1}{\Delta x} & & & & & & & & \\
& \frac{1}{\Delta x} & & & & & & & \\
-\frac{1}{\Delta x} & & \ddots & & & & & & \\
& & & \frac{1}{\Delta x} & & & & & \\
& & & & \frac{1}{\Delta x} & & & & \\
& & & & & \ddots & & & \\
\hline
& & & & & & \ddots & & \\
& & & & & & & -\frac{1}{\Delta x} & \frac{1}{\Delta x} \\
& & & & & & & -\frac{1}{\Delta x} & \frac{1}{\Delta x}
\end{array}\right)
\begin{pmatrix} U_{11} \\ U_{12} \\ \vdots \\ U_{1n} \\ U_{21} \\ \vdots \\ U_{2n} \\ U_{31} \\ \vdots \\ \vdots \\ U_{mn} \end{pmatrix}
$$

$$
\frac{\partial U}{\partial y}
\begin{pmatrix}
\begin{matrix}
\frac{1}{\Delta y} & & \\
-\frac{1}{\Delta y} & \frac{1}{\Delta y} & \\
& \ddots & \ddots \\
& -\frac{1}{\Delta y} & \frac{1}{\Delta y}
\end{matrix} & & & \\
& \begin{matrix}
\frac{1}{\Delta y} & & \\
-\frac{1}{\Delta y} & \frac{1}{\Delta y} & \\
& \ddots & \ddots \\
& -\frac{1}{\Delta y} & \frac{1}{\Delta y}
\end{matrix} & & \\
& & \ddots & \\
& & & \begin{matrix}
\frac{1}{\Delta y} & & \\
-\frac{1}{\Delta y} & \frac{1}{\Delta y} & \\
& \ddots & \ddots \\
& -\frac{1}{\Delta y} & \frac{1}{\Delta y}
\end{matrix}
\end{pmatrix}
\begin{pmatrix}
U_{11} \\ U_{12} \\ \vdots \\ U_{1n} \\ U_{21} \\ \vdots \\ U_{2n} \\ U_{31} \\ \vdots \\ \vdots \\ U_{mn}
\end{pmatrix}
$$

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