Problem Set 1 Solution
MEMORANDUM

16.851 Satellite Engineering

To: Professor David W. Miller and John E. Keesee

From: Students

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Subj: Potential Orbits about the Lagrangian Points

cc: Marilyn Good

MOTIVATION

In the 1960's Robert W. Farquhar, with his thesis advisor Professor John Breakwell, tackled a NASA problem involving a spacecraft orbiting about the Earth-Moon L2 Lagrange point. The goal was a communications satellite to service the backside of the Moon, with an uninterrupted view of both the Earth and the far side of the Moon. Even though they were not the first to tackle orbits at the Lagrange points, their work has provided a foundation for modern missions about these "stationary" points, and has motivated the use of these points for current and future missions.

Currently, one of NASA's long term objectives for manned space travel is to expand mission capability from Low Earth Orbit to anywhere in the solar system[8]. A major stepping stone to exploring the solar system is developing space observatories at the Lagrangian Points. The Lagrangian Points have been used in the past for systems such as the International Sun-Earth Explorer-3 (ISEE-3) and Solar and Heliospheric Observatory Satellite (SOHO). The ISEE-3 spacecraft observed the Earth from a halo orbit at the Sun-Earth L1 point, while SOHO observed the *Sun* from the same point. From the L1 point, SOHO has the unique ability to maintain continuous line-of-site to the Sun (no eclipse), while also being continually able to return data to the Earth. Also several future missions plan on using these strategic points for telescopes, moon communication satellites, and observatories. For example, a mission has been proposed to place space telescopes at each of Jupiter's Lagrange points L4 and L5 for cooperative, ultra-long-baseline astronomical observations[2]. Understanding how these unique locations can be exploited for space exploration becomes a necessary task.

PROBLEM STATEMENT

As future space missions plan to utilize Lagrange Point Orbits, describe the potential orbits about the stable and unstable Lagrange points.

APPROACH

In order to answer this question, background information must be provided on the nature of the Lagrange Points. A literature search will supplement SMAD[3] and yield a description of the Lagrange Points, their stability, potential orbits around them, and further motivations for Lagrange Point missions. Analysis based on the literature study will allow the creation of software tools to predict and visualize the locations of the Lagrange Points for a given two-body system, as well as the determination of maneuver requirements to maintain Lagrange Point Orbits.

SOLUTION

Perspective

Joseph Louis Lagrange (1736-1813) showed that, given two large bodies (i.e. the Earth and the Moon) in circular orbits about each other, there exist 5 points where an object could remain stationary with respect to the two major bodies, as shown in Figure 1. Lagrange further showed that the points will exist for any pair of orbiting bodies as long as the mass ratio between them is larger than 25. While some of these "Lagrange Points" were shown to be unstable, a spacecraft could potentially orbit any of them with minimal control effort. The points are also known as the "equilibrium" or "libration" points, because at any one of these points, the gravitational attraction of the two bodies on a spacecraft and the spacecraft's centrifugal acceleration are balanced. Generally, when analyzing the Lagrangian points, the origin of the reference frame is at the barycenter of the two bodies and rotates with an angular velocity equivalent to the period of the co-orbiting bodies. (For the Earth-Moon system, the barycenter is located inside the Earth at 4,700 *km* toward the moon[4] and the angular velocity is approximately the period of the Moon about the Earth ~ 1 month.) All five of these strategic points exist in the plane of motion.
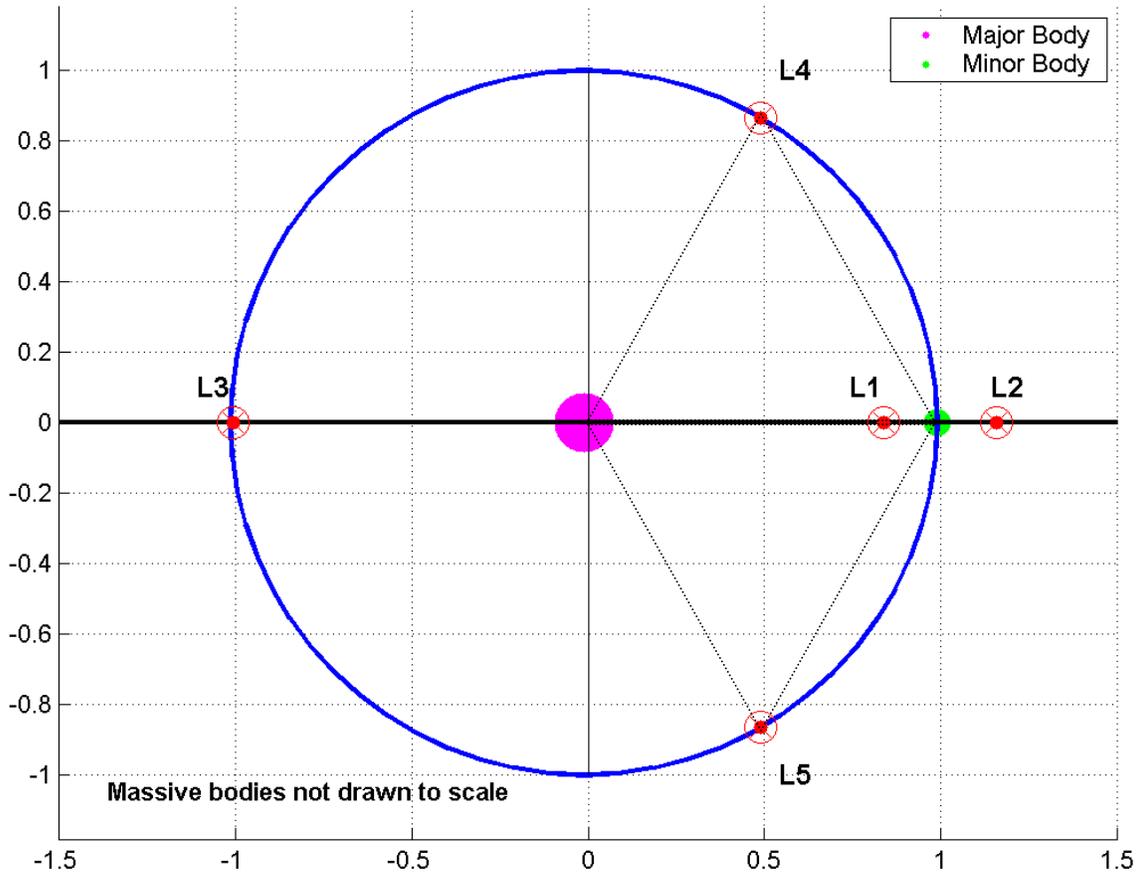
Figure 1. Locations of the Lagrange Points for an arbitrary 2-body system

The three points located on the line which connects the two bodies (L1, L2, L3) are referred to as the collinear points, and are determined from Lagrange's famous Quintic Equation, presented by Richard Battin[5] as:

$$(m_1 + m_2)\chi^5 + (3m_1 + 2m_2)\chi^4 + (3m_1 + m_2)\chi^3$$
$$- (m_2 + 3m_3)\chi^2 - (2m_2 + 3m_3)\chi - (m_2 + m_3) = 0$$

(Eq. 1)

where m_1, m_2 and m_3 represent the masses of the three bodies (large body, small body, and spacecraft). The mass of the spacecraft is assumed to be zero when solving the equation. Depending on the location of the spacecraft there are 3 different solutions for the Quintic Equation (when m_1=0, m_2=0, or m_3=0) Though not clear at first, the term χ is used to calculate the three different collinear points as shown in Figure 2.

$L_1 = \rho$ = Unit Distance/ $(1+\chi)$

$L_2 = \rho(1+\chi)$ = Unit distance $(1+\chi)$

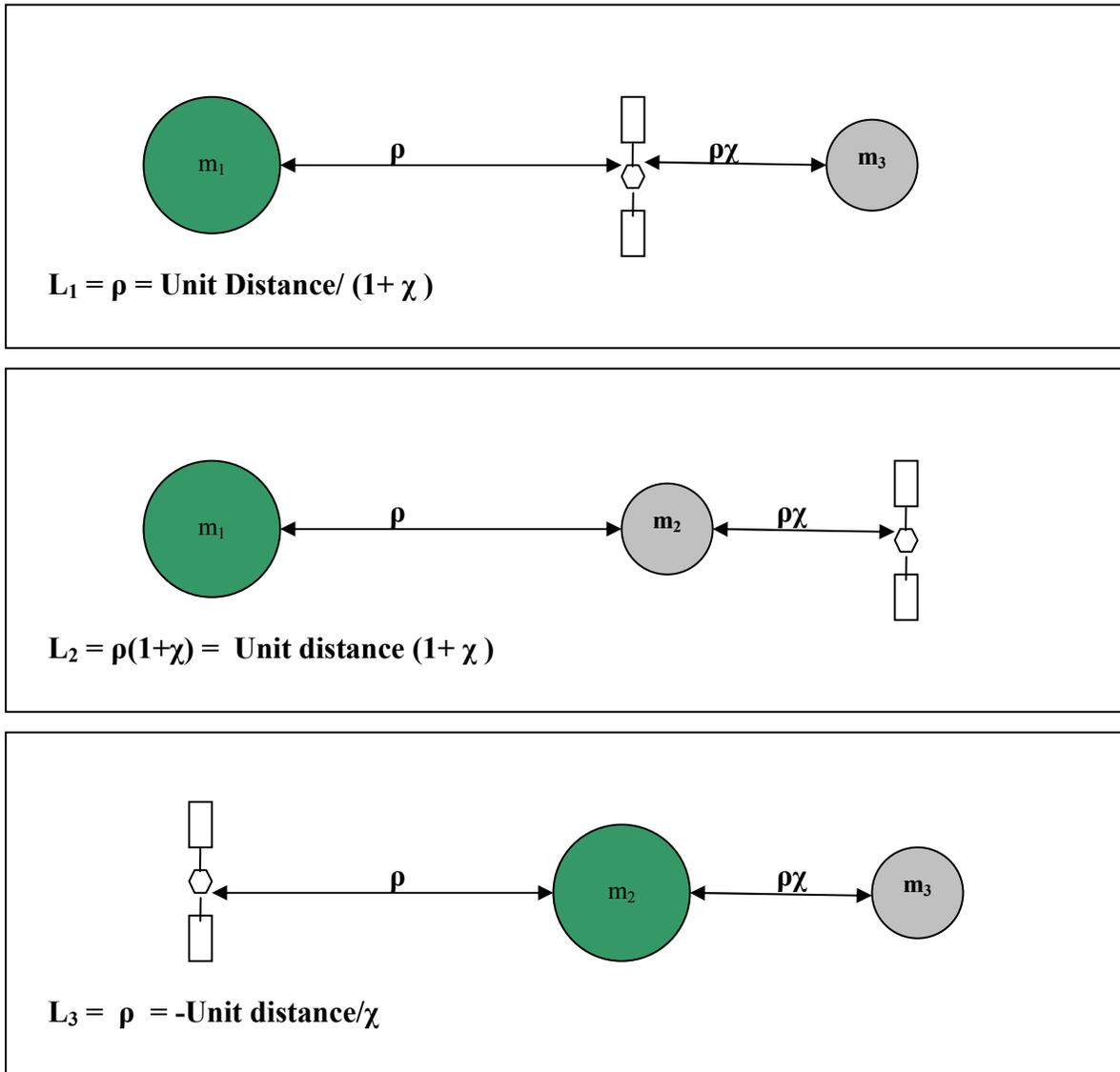**$L_3 = \rho$ = -Unit distance/χ**

Figure 2. Graphical interpretation of the Quintic Equation

These points can also be looked at as a balance of forces such that objects in these locations have a period equivalent to the co-orbiting bodies. To illustrate, if the smaller body was not present, then a spacecraft located at these particular points would have an orbital period either greater than or less than the co-orbiting period. However, due to the presence of the smaller body, gravitational forces act on the objects in these regions, such that their orbital period either increases or decreases to be equivalent to the smaller body. This produces the effect that (in the relative, rotating frame) objects located at these points remain stationary with respect to the two bodies[4].

The L4 and L5 points are referred to as the Triangular Points. This is because they are at the points of an equilateral triangle, where the base of the triangle is the line between the two bodies, as shown in Figure 1. Without the presence of the smaller body, the net gravitational force is directed toward the center of the Earth. However, the gravitational force due to the smaller body causes the total gravitational force to point toward the barycenter of the system and produce a period of revolution equivalent to the co-orbiting bodies[4].

Stability and Langrage Orbits

The three collinear points, L1, L2, and L3, are considered unstable and are like "saddles" in the gravitational potential, while the two triangular points are stable. In the plots of the gravitational potential below, the large peak represents the gravity potential near the large body, and the smaller peak represents the gravity potential near the small body. Notice the "saddle" where the L1 point is located. The origin of these plots is explained in the Analysis section, below.
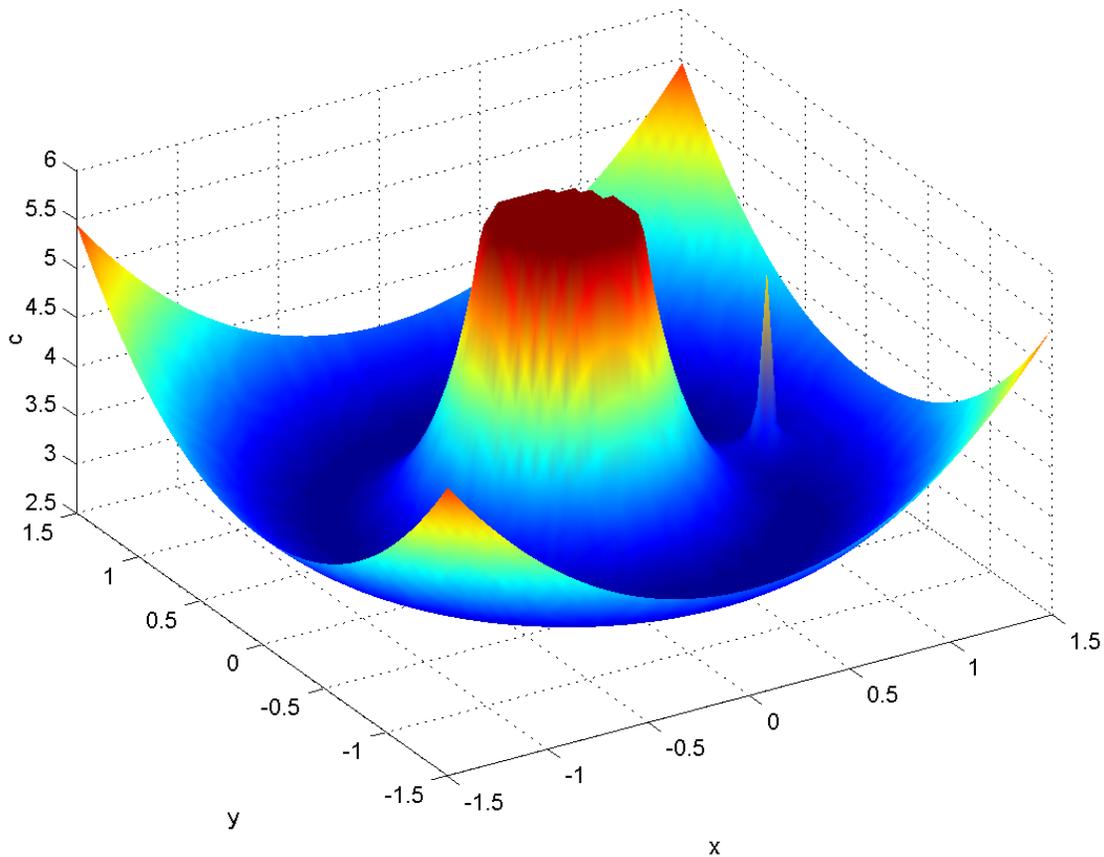

Figure 3. 3D plot of gravitational potential in the rotating reference frame
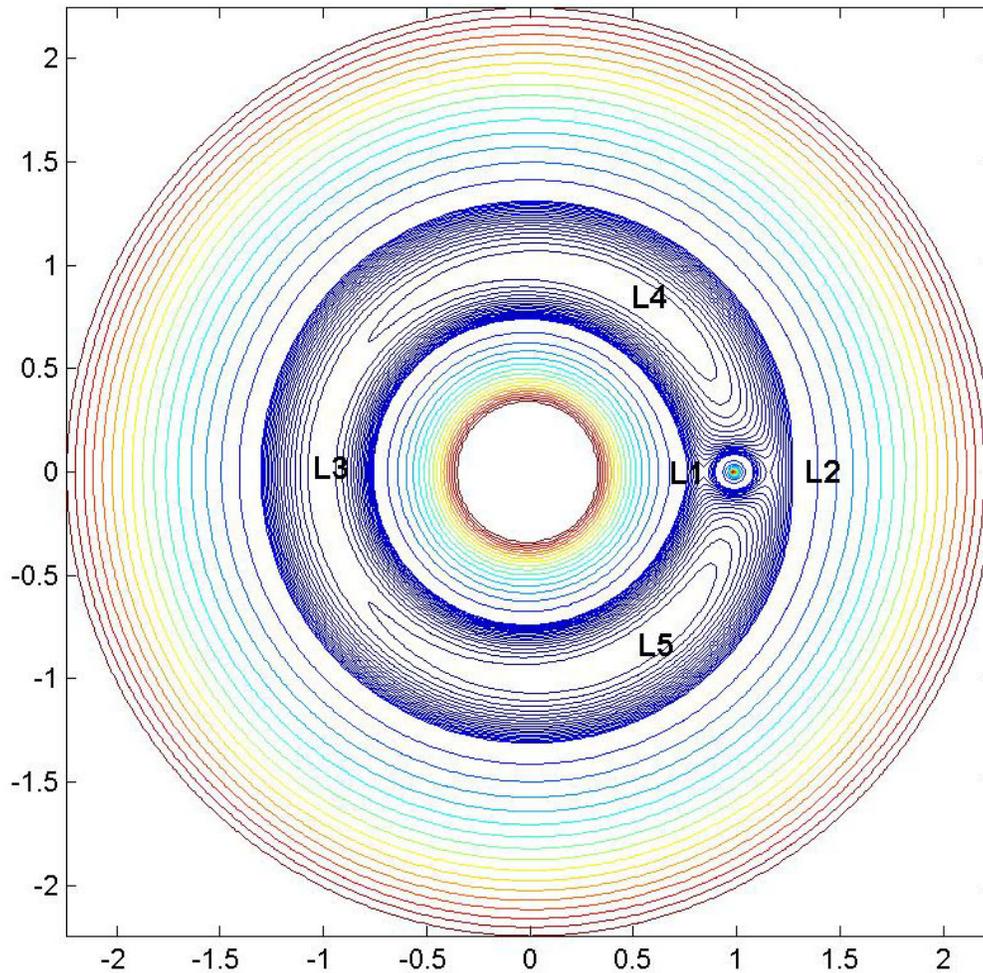
Figure 4. 2D contour plot of gravitational potential in the rotating reference frame

It has been proven in theory and in application that there are periodic orbits about the unstable Lagrange points (no special treatment is required for the stable L4 and L5 points, once there). Since there is no closed form solution for the equations of these orbits, computational and numerical analysis are used to determine the trajectories. Also, these orbits can not be described by specific orbital parameters (orbital elements) as is common for typical orbits, but there are "families" of orbits that exist. "As early as 1963, Boudas computed 19 families of three-dimensional periodic orbits in the circular restricted three-body problem."[6] Though there are many names and different methods of organization, there are 3 main categories in present literature (note that not all orbits can be classified into one of these types, however these are the most practical and well-studied orbits):

1. **Lyapunov Orbit**
    The Lyapunov Orbit is a periodic orbit in the plane of motion of the primary bodies.

2. **Lissajous Orbit**
   The Lissajous Orbit is a combination of a planar and vertical components in the periodic orbit (see Figure 6)

3. **Halo Orbit**:
   The Halo Orbit is a special case of Lissajous orbits where the in-plane and out-of-plane frequencies are equal[11,6].

Equations and Analysis

Much work has been done to analyze the motion of satellites in the vicinity of the Lagrange points. As mentioned above, Robert W. Farquhar wrote his PhD thesis on this topic under a NASA grant, and this technical report[1] has become the bibliographical root of almost every Lagrange Point paper since 1969 (as it is the root of this one). In the first section of his work, Farquhar derives the equations of motion for the *circular restricted three-body problem* (CR3BP), which can be used to plot both Lagrange Point orbits and the transfer orbits required to get there. This formulation is characterized by a rotating coordinate system with an origin at the center of mass of the two large bodies (the barycenter). The x-axis of the coordinate system is along the line connecting the primary bodies, the z-axis in the direction of the angular momentum vector of the system, and the y-axis completing the right handed set. The coordinate frame thus rotates with the angular velocity of the system. This arrangement is illustrated in Figure 1.

Several other assumptions are important to the circular restricted three-body equations, most of them evidenced by the name itself. First, the orbit of the small body (for example, the Moon in the Earth-Moon system) is assumed to be circular. This is a reasonable assumption for most systems of interest, including Earth-Moon[9] (e_m=0.055), Sun-Earth[7] (e_e=0.0167), and Sun-Jupiter[7] (e_j=0.0482). The term *restricted* refers to the assumption that the third body does not affect the orbits of the two primary bodies. That is, the third body is massless. Another assumption is that the universe is strictly limited to the three bodies under consideration, so there are no other gravitational influences in the system. For example, it is assumed that the Sun does not affect the orbit of a spacecraft about an Earth-Moon Lagrange Point. Finally, it is assumed that there are no other unmodeled accelerations in the system, such as solar radiation pressure. All of these assumptions, while seemingly very restrictive, allow the formulation of the equations of motion that give a representative first-order picture of orbits about the Lagrange Points. While not suitable for a real spacecraft trajectory, the orbits that can be generated with the CR3BP equations are useful from an initial design perspective.

There are many equivalent formulations of the CR3BP equations as presented by different authors. One of the most readable forms is given by Paul D. Merritt[9] and is presented here (Farquhar's equations are similar, using different notation):

$$\ddot{x} = 2\dot{y} + x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu[x-(1-\mu)]}{r_2^3} \qquad \text{(Eq. 2)}$$

$$\ddot{y} = -2\dot{x} + y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} \qquad \text{(Eq. 3)}$$

$$\ddot{z} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3} \qquad \text{(Eq. 4)}$$

In the above equations, x, y, and z represent the position of the satellite with respect to the barycenter, where r_1 is the distance from the large body to the satellite, r_2 is the distance from the small body to the satellite, and μ is the nondimensional center of mass:

$$\mu = \frac{M_2}{M_1 + M_2} \qquad \text{(Eq. 5)}$$

Here M_1 is the larger body. It is important to note also that, in the normalized unit system of these equations, the angular rate of the two bodies about the barycenter has been set to unity, resulting in a unit of time (TU) that is the inverse of the mean angular rate. That is:

$$\dot{\theta} = 1 \qquad \text{(Eq. 6)}$$

$$1\,TU = \frac{1}{n} = \frac{P}{2\pi}\,s \qquad \text{(Eq. 7)}$$

Where n is the mean angular rate, or P is the period of the masses about the barycenter. The equations above (Eq. 2-4) are a system of three coupled nonlinear differential equations that can be used to model the motion of a spacecraft on orbits around the Lagrange points. No closed-form solution exists to these equations, even with the simplifying assumptions listed above. Given the proper initial conditions, however, a designer can use a numerical integration routine to plot a reference trajectory for initial analysis. Such a tool has been created for this report, and is presented in the following section.

Unfortunately, deriving the necessary initial conditions is no trivial task. A simple example can be formed using the stable L4 and L5 points. Since the location of these points is known precisely, and no critical initial velocity is required, a sample spacecraft can be inserted into the L4 or L5 point and observed for an extended time. Using the tool mentioned above, this experiment was performed, and the results are as expected. When inserted precisely at the L4 or L5 point with no initial velocity, the spacecraft *has no relative motion at all* for extended periods. The result is a plot with a single point at the L4 and L5 points, and so is not presented here. A more interesting example is formed when the spacecraft is *not* put exactly at the Lagrange point (or is put there with nonzero relative velocity), and is allowed to obey the mechanics of the CR3BP for an extended period. The result of this experiment is shown in Figure 5. Compare the resulting pattern to Figures 4 and, and then the CR3BP equations may be used with confidence.
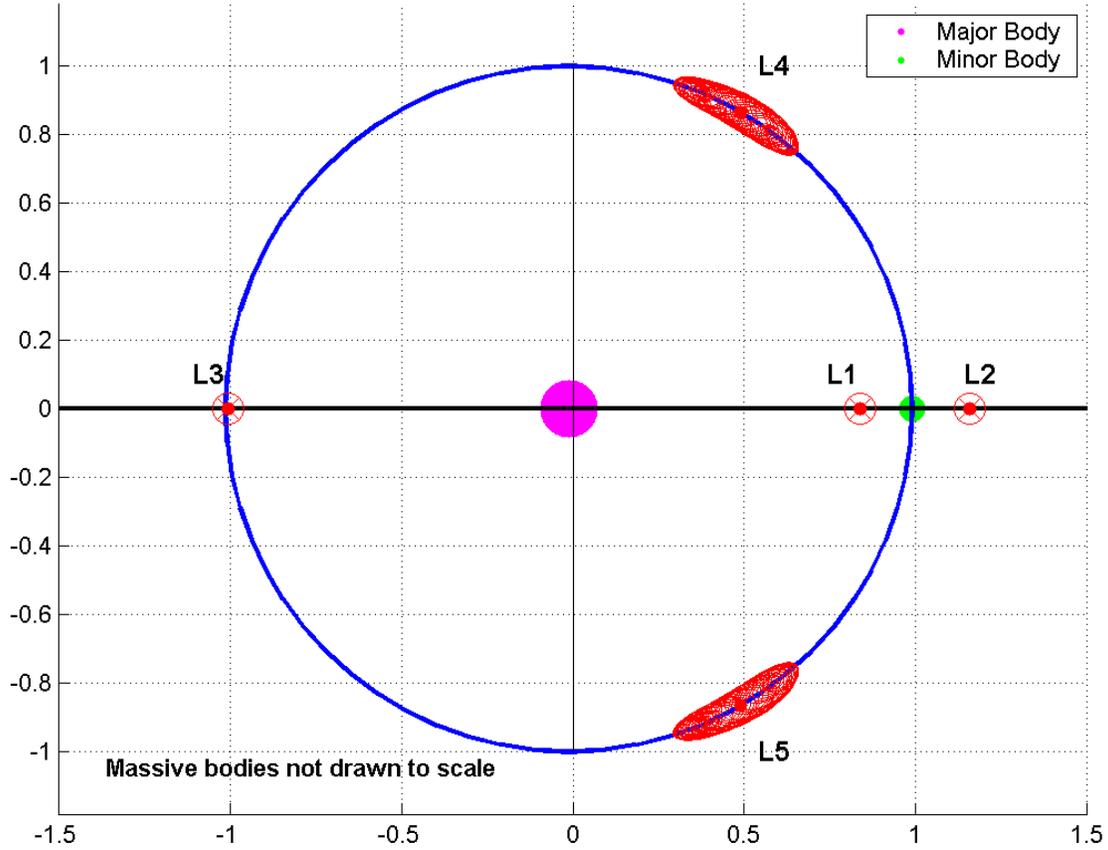
Figure 5. Perturbed and bounded motion at the L4 and L5 Lagrange Points

Research has indicated several methods that have been used in the past to generate initial conditions (and even full trajectories), including sophisticated software tools developed by industry or educational institutions. One such software tool is called Generator and was created by Purdue University, as described by Hamilton, et. al[10]. Generator uses ephemeris data to model perturbations not included in the CR3BP model, including eccentricities, external gravitational influences, and radiation pressure. Another method, presented clearly by Wang Sang Koon[11] of Caltech, derives analytic periodic solutions to linearized versions of the CR3BP. These equations can be used as a first approximation to the desired halo orbit, for use in more advanced propagation and correction algorithms for Lagrange Point orbits. These periodic solutions for the linearized problem are as follows:

$$x = -A_x \cos(\lambda t + \phi) \qquad \text{(Eq. 8)}$$

$$y = kA_x \sin(\lambda t + \phi) \qquad \text{(Eq. 9)}$$

$$z = A_z \sin(\nu t + \psi) \qquad \text{(Eq. 10)}$$

Here A_x, $A_y=kA_x$, and A_z are amplitudes that characterize the orbit. The parameters λ and v are the frequencies of the orbit in the respective axes, and φ and ψ are phase angles. If λ and v are unequal, the result is a Lissajous orbit; if they are equal, the result is a halo orbit. A sample plot generated by these equations is shown in Figure 6. It should be noted that the origin of this plot is at the L1 point based on the formulation of Eqs. 8-10.
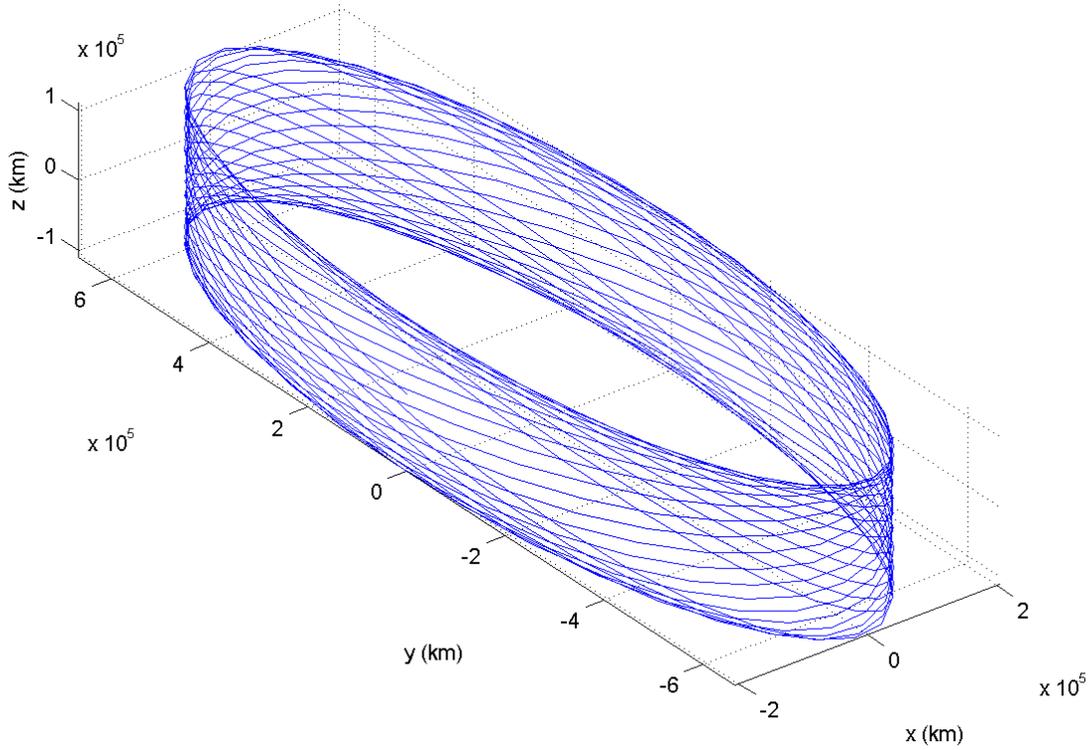

Figure 6. 3D plot of a Lissajous orbit about L1, using solution from linearized CR3BP

In addition to the equations of motion for the restricted three body problem, Farquhar derives in Chapter 1 of his thesis the equation for the Jacobian constant and the surfaces of constant velocity. Using energy methods and drawing upon the equations for the three body problem, the following equation can be derived:

$$V^2 = (x^2 + y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} - C \qquad \text{(Eq. 11)}$$

Here V is the magnitude of the relative velocity of the satellite in the rotating frame, x and y are the position of the satellite in the same frame, C is the Jacobian constant, and μ is as defined above in Eq. 5. Of course this equation only describes motion in the orbital plane. By setting the relative velocity to zero, a contour map can be created that indicates lines of constant C. This map is the potential field of the given system as viewed in the rotating coordinate frame, as presented in Figures 3 and 4.

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Orbit Maintenance Requirements

Initially there was an expectation that a software tool could and would be created, that would estimate maneuver requirements for Lagrange point orbits. Knowing initially that orbits about the Lagrange Points were unstable, and periodic maintenance was thus required (regardless of perturbing forces), it was expected that a simple algorithm could be found or derived to give the designer a first cut at the ΔV budget for his mission. The result of research in this area was surprising, and the immediate consequence is that the designer does not need a software tool to make a basic ΔV estimate for the mission. The secondary consequence is that mission designers need quite sophisticated software tools to obtain a high-fidelity model of the sizes and frequencies of the orbit maintenance maneuvers. An explanation for both of these consequences follows, beginning however with the methods that exist for calculating the requirements.

Regardless of the perturbing forces involved, orbits about the collinear Lagrange Points are unstable and thus require periodic maintenance. The three body problem, however, even in its simplest and most restricted form, still results in a system of three coupled nonlinear differential equations. Farquhar himself discussed possible station-keeping strategies in his original work[1], and later advanced this effort in at least one more paper published in 1980[12]. Farquhar's original work dealt with continuous thrust techniques (and indeed even applied solar sails to this application), while many of the techniques to follow dealt in discrete burns suitable for chemical propulsion devices. Hamilton[10] used a discrete linear-quadratic-regulator framework to control the orbit of spacecraft at the Sun-Earth L2 point, with applications for formation flying at that location. Gomez, et. al.[13], wrote a paper detailing two methods of control, called the Target Point strategy and the Floquet Mode approach. The Target Point strategy computes maneuvers designed to keep the spacecraft near a reference orbit, using a cost function that incorporates required control energy and predicted deviation from the nominal (based on the calculated maneuvers). The Floquet Mode approach is a sophisticated system based on the linearized equations of the CR3BP described above (Eq. 8-10). Serban, Koon, et. al., in a much more recent paper[14], have used optimal control to generate halo orbit maneuver correction strategies, with additional emphasis on transfer trajectories to halo orbits.

Unfortunately, none of these methods has a simple implementation that could be quickly and neatly coded into a useful design tool for satellite engineers. Fortunately, the result of all this effort in control strategies yields a consistent and valuable generalization: orbits around the Lagrange Points can be maintained for years using a very small amount of ΔV, and with relatively infrequent maneuvers [see 13, 15, 16]. A *conservative* rule-of-thumb for designers would be an estimate of 12 maneuvers per year, with a total ΔV of only 20 *m/s* per year. Examples have been found with as few as 6 maneuvers, and ΔV requirements only 4 *m/s* per year.

Software Description

A number of software tools were written in the course of the preceding analysis. All functions were written in Matlab, and their dependencies and flow (where applicable) are indicated in Figure 7. The following text is a brief description of each function and its

intention, and then the commented code follows as an Appendix. The code is documented such that, with the help of this description, a user could immediately begin using the tools to generate useful visualizations.
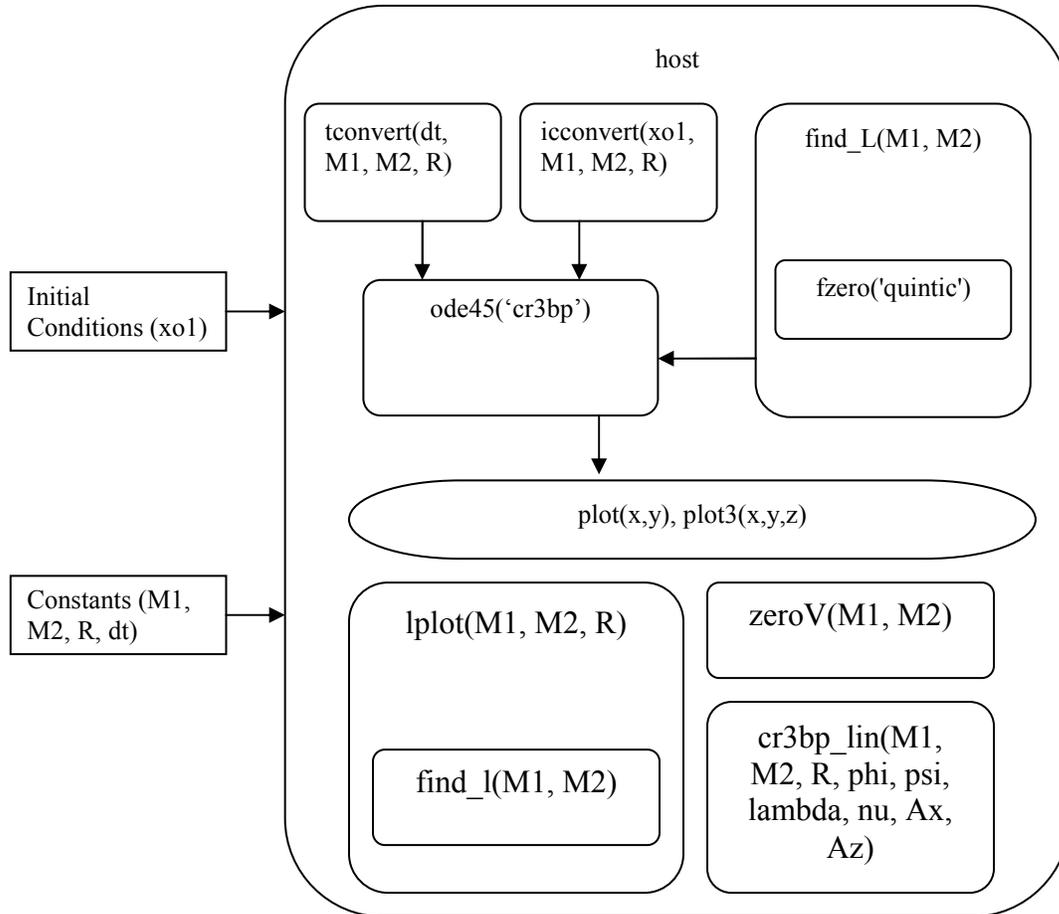

Figure 7. Dependencies and flow diagram for analysis functions

The primary code, *host.m*, is a host program that calls all of the others. It is here that the user can specify the desired parameters such as: the mass of the primary bodies (*M1, M2*), the distance between them (*R*), the run time of the simulation (*dt*), the initial conditions of the orbits (*xo1*), and the parameters of the linearized equations of the three body problem (*phi, psi, lambda, nu, Ax, Az*).

Because all of the equations from the previous section are formulated in normalized units, the conversion programs *tconvert.m* and *icconvert.m* were created to convert standard units of *seconds*, *meters*, and *meters/second* into *TU*, *DU*, and *DU/TU*, respective. These functions can be called in-line with the other functions, so the conversion is transparent to the user.

The function *find_l.m* is useful for several purposes. First, as seen in the diagram, it can be used to create or supplement the initial conditions for the equations of motion. For example, this function was used to locate the L4 and L5 points (and then these points were offset a bit to make the plot interesting) for the initial conditions to the plots in Figure 5. Secondly, the output from the function is used to create the 2D visualization of the Lagrange Points for any system, as shown in Figure 1. Function *find_l.m* works by numerically solving Lagrange's quintic equation for the three collinear points, and then calculating the remaining two points with simple trigonometry.

The function *cr3bp.m* is describes the three coupled nonlinear differential equations of motion for the CR3BP. The function is called using one of Matlab's built-in numerical integrators, such as *ode45.m*, to determine the position and velocity of the spacecraft in time. The output from *cr3bp.m* can be plotted in two or three dimensions using Matlab's built-in plot commands. Again, a two-dimensional example of this function's capability is shown in Figure 5.

The function *lplot.m*, as mentioned above, is used to create the two-dimensional visualization of the Lagrange points in any system, as shown in Figure 1. The generated plot has its origin at the barycenter of the two primary bodies.

The function *zerov.m* implements Eq. 11 and creates both two- and three-dimensional representations of the potential field of the two central bodies, in the rotating coordinate frame.

Finally, the function *cr3bp_lin.m* is an implementation of Eqs. 8-10, and generates two- and three-dimensional plots of the periodic solution to the linearized CR3BP. The function requires many inputs that describe the amplitude and frequency of the desired orbit in all three axes.

APPENDIX

Software Code

host.m

```
global M1    % must be global to work in ode45 call
global M2
global M3

M1=5.9736e24;      %kg, earth
M2=7.35e22;        %kg, moon
R=3.84e8;          %m, earth-moon
% M1=1.9891e30;    % kg, sun
% M2=5.9736e24;    % kg, earth
% R=1.50e11;       % m, earth-sun
dt=1800*24*60*60;  % s, simulation time

points=find_l(M1, M2);

xo1 = [.5 -sqrt(3)/2 0 0 0 0]';   % IC's for oscillation at L5/L4

% If t needs to be converted, use 'tconvert' in the form "tconvert(dt, M1, M2, R)"
% If the IC's need conversion, use 'icconvert' in the form "icconvert(xo1, M1, M2, R)"
% Ex: [t y]=ode45('cr3bp', [0 tconvert(dt, M1, M2, R)], icconvert(xo1, M1, M2, R));

[t y]=ode45('cr3bp', [0 tconvert(dt, M1, M2, R)], xo1);   % here xo1 is already
barycentered

%%%%%% make 2D plot of the CR3BP %%%%%%
figure
plot(y(:,1),y(:,2))
xlabel('x')
ylabel('y')
grid on
axis equal
%%%%%% /make 2D plot of the CR3BP %%%%%%

%%%%%% make 3D plot of the CR3BP %%%%%%
figure
plot3(y(:,1),y(:,2),y(:,3))
grid on
xlabel('x')
ylabel('y')
zlabel('z')
axis equal
%%%%%% /make 3D plot of the CR3BP %%%%%%

%%%%%% make plots of the surfaces of zero velocity %%%%%
zerov(M1, M2, 6)
%%%%%% /make plots of the surfaces of zero velocity %%%%%

%%%%%% make the 2D plot of L-point locations %%%%%%
lplot(M1, M2)
%%%%%% /make the 2D plot of L-point locations %%%%%%


%%%%% plot the periodic solutions to linearized CR3BP %%%%%

% Here, because of the example data available, this example is called
%   for the sun-earth system

Ax = 206e6;
Az = 110e6;
cr3bp_lin(1.9891e30, 5.9736e24, 1.50e11, Ax, Az, 0, 0, 2.086, 2.015);
%%%%% plot the periodic solutions to linearized CR3BP %%%%%
```

tconvert.m

```
function tout=tconvert(t, Mb, Ms, r)

% tout is the converted time in units of TU for the given system
% t is the time to convert from, in units of seconds
% Mb is large body
% Ms is the small body
% r is the distance between them

G = 6.67e-11;  % N m^2 / kg^2

P = sqrt( (4*pi*pi*r*r*r) / (G*(Mb+Ms)) );   % s, Period of small body's rotation about
barycenter

tout = (2 * pi * t) / P;     % the time in TU
```

icconvert.m

```
function ic_out=icconvert(x, Mb, Ms, r)

% ic_out is the converted initial state in units of DU and DU/TU for the given system
% x is the state to convert from, in units of meters and meters/second
% Mb is large body in kg
% Ms is the small body in kg
% r is the distance between them in meters

G = 6.67e-11;  % N m^2 / kg^2

P = sqrt( (4*pi*pi*r*r*r) / (G*(Mb+Ms)) );   % s, Period of small body's rotation about
barycenter

TU = P / (2 * pi);     % 1 TU, in seconds

ic_out(1:3,:) = x(1:3,:) / r;          % initial conditions converted to DU
ic_out(4:6,:) = x(4:6,:) * TU / r;     % initial conditions converted to DU/TU
```

```
function l_points=find_l(Ml, Ms)

% Function "l_points" find the 5 lagrange points for a given system, in
%   normalized units (assuming the distance between the bodies is 1)
%
% l_points is returned as a length 5 vector of the lagrange points, in order
%   Lagrange points will be referenced from the barycenter, + towards small body
%   All z-components are assumed to be zero, planar system
% Ml is large body in kg
% Ms is the small body in kg

global Ma      % must be global to get these variables to 'quintic' through fzero
global Mb
global Mc

r = 1;   % the distance from the big body to small body in normalized units

%%%%%%%%  Solve for L1  %%%%%%%%%%
Ma=Ml;
Mb=0;
Mc=Ms;

l_points(1,1)=r/(fzero('quintic', .1) + 1) - Ms/(Ml+Ms);
l_points(1,2)=0;
%%%%%%%%  /Solve for L1  %%%%%%%%%%


%%%%%%%%  Solve for L2  %%%%%%%%%%
Ma=Ml;
Mb=Ms;
Mc=0;

l_points(2,1)=r*fzero('quintic', .1) + 1 - Ms/(Ml+Ms);
l_points(2,2)=0;
%%%%%%%%  /Solve for L2  %%%%%%%%%%


%%%%%%%%  Solve for L1  %%%%%%%%%%
Ma=0;
Mb=Ml;
Mc=Ms;

l_points(3,1)=-r/fzero('quintic', .1) - Ms/(Ml+Ms);
l_points(3,2)=0;
%%%%%%%%  /Solve for L3  %%%%%%%%%%



%%%%%%%%  Solve for L4 & L5  %%%%%%%%%%
l_points(4,:)= [.5-(Ms/(Ml+Ms)) sqrt(3)/2];
l_points(5,:)= [.5-(Ms/(Ml+Ms)) -sqrt(3)/2];
    % these are simply the corners of the equilateral triangle where one
    %   side is the distance between the two primary bodies
%%%%%%%%  /Solve for L4 & L5  %%%%%%%%%%
```

cr3bp.m

```
function xdot = cr3bp(t, x);

% function cr3bp implements the CR3BP equations of motion

% x is the state vector [r v], [6x1]

global M1;
global M2;

MU = M2 / (M1 + M2);

r_eb_mag = 1 * MU;          % distance from Big body to Barycenter, in normalized
distance units (MDU)
r_mb_mag = 1 - r_eb_mag;    % distance from Small body to Barycenter (MDU)

r_eb = r_eb_mag * [ 1 0 0]';    % vector from Big body to Barycenter (MDU)
r_mb = r_mb_mag * [-1 0 0]';    % vector from Small to Barycenter (MDU)

xdot(1:3,1) = x(4:6,1);
rho = x(1:3,1);         % barycenter to satellite
v = x(4:6,1);

r1 = r_eb + rho;        % Big body to satellite
r2 = r_mb + rho;        % Small body to satellite

r1_mag = norm(r1);
r1_3 = r1_mag * r1_mag * r1_mag;

r2_mag = norm(r2);
r2_3 = r2_mag * r2_mag * r2_mag;

xdd =  2 * v(2) + rho(1) - ((1-MU)*(rho(1)+MU))/r1_3 - MU*(rho(1)-(1-MU))/r2_3;
ydd = -2 * v(1) + rho(2) - ((1-MU)*(rho(2)    ))/r1_3 - MU*(rho(2)        )/r2_3;
zdd =                    - ((1-MU)*(rho(3)    ))/r1_3 - MU*(rho(3)        )/r2_3;

xdot(4:6,1) = [xdd ydd zdd]';
```

lplot.m

```
function lplot(Ml, Ms)

r_sun = .1;     % scaled diameter of the large body (i.e. sun), not to scale
r_earth = .05;  % scaled diameter of the small body (i.e. earth), not to scale

points=find_l(Ml, Ms);

MU = Ms / (Ml + Ms);    % large body to barycenter in normalized distance units

figure
hold on
theta = 0:pi/32:2*pi;
plot(r_sun*sin(theta)-MU, r_sun*cos(theta),'m')
plot(r_earth*sin(theta)+1-MU, r_earth*cos(theta),'g')
legend('Major Body','Minor Body')
plot(sin(theta)-MU, cos(theta),'b')
x_axis=-1.5:1.5;
y_axis=-1:1;
plot(x_axis, zeros(1,length(x_axis)),'k','linewidth',2)
plot(zeros(1,length(y_axis)),y_axis,'k')
plot(points(:,1),points(:,2),'rx','markersize',12)

text(-1.4, -1.05, 'Massive bodies not drawn to scale')

axis equal
grid on
```

```
function zerov(Ml, Ms, max_height)

% function zerov plots the 2D and 3D representations of the
%   potential near Ml and Ms in the rotating coordinate frame

% Ml is large body in kg
% Ms is the small body in kg
% "max_height" is where the top of the 3D plot is cut off, otherwise
%    there are scaling issues and the plot is not useful

clear c

% M1 = 5.97e24;  % kg, Mass of Earth
% M2 = 7.35e22;  % kg, Mass of Moon


MU = Ms / (Ml + Ms);

r_eb_mag = 1 * MU;            % distance from Earth to Barycenter, in Moon distance units
(MDU)
r_mb_mag = 1 - r_eb_mag;     % distance from Moon to Barycenter (MDU)

r_eb = r_eb_mag * [ 1 0 0]';    % vector from Earth to Barycenter (MDU)
r_mb = r_mb_mag * [-1 0 0]';    % vector from Moon to Barycenter (MDU)

% x=-3:.02:3;
% y=-3:.02:3;
x=-1.5:.05:1.5;           % the x-plot range
y=-1.5:.05:1.5;           % the y-plot range

for i=1:length(x)
    for j=1:length(y)
        rho=[x(i) y(j) 0]';

        r1 = r_eb + rho;       % big body to satellite
        r1_mag = norm(r1);
        r2 = r_mb + rho;       % small body to satellite
        r2_mag = norm(r2);

        c(j,i) = x(i)^2 + y(j)^2 + 2*(1-MU)/r1_mag + 2*MU/r2_mag;  % zero v_rel equation
        if c(j,i)>max_height
            c(j,i)=max_height;
        end
    end
end


%  The following manipulations create meaningful contour lines on the
%       contour plot
csize=size(c);
data=reshape(c,[1 csize(1)*csize(2)]);
[numbinned centers]=hist(data,18);
low_bin=centers(1);
min_el=min(data);
max_el=max(data);
tmp=log(centers)/log(15);
contourlist=[sort([tmp+(min_el-tmp(1)) centers(find(centers>max(tmp+(min_el-
tmp(1)))))])];
%  End manipulations


figure
[a b]=contour(x,y,c,contourlist);
axis equal
xlabel('x');
ylabel('y');

figure
[a b]=contour3(x,y,c,contourlist);
axis equal
xlabel('x');
ylabel('y');
zlabel('c');

figure
```

```
surf(x,y,c);
xlabel('x');
ylabel('y');
zlabel('c');
```

cr3bp_lin.m

```
function cr3bp_lin(Ml, Ms, R, Ax, Az, phi, psi, lambda, nu)

% function cr3bp_lin plots the periodic orbits based on the
%   linearized CR3BP equations

% Ml is large body in kg
% Ms is the small body in kg
% R is the distance between them in meters
% phi and psi are phase angles of the orbit
% lambda and nu are frequencies of the orbit (radians)
% Ax and Az are the amplitudes of the orbit (in meters)

MU = Ms / (Ml + Ms);      % large body to barycenter in normalized distance units
                          %    or unitless center of mass

points=find_l(Ml, Ms);    % normalized distance

gamma=1-points(1,1)-MU;   % normalized distance units

c2 = (gamma^-3)*(MU + ((-1)^2)*(1-MU)*(gamma/(1-gamma))^3);   % unitless

k = (lambda^2 + 1 + 2*c2)/(2*lambda);

Ax = Ax/R;
Ay = (k*Ax);
Az = Az/R;

t=0:.1:80;

x =  -Ax * cos(lambda.*t + phi);      % the solutions of the linearized eqs
y =   Ay * sin(lambda.*t + phi);
z =   Az * sin(nu    .*t + psi);
xd =  lambda * Ax * sin(lambda.*t + phi);  % and their derivatives, if required
yd =  lambda * Ay * cos(lambda.*t + phi);
zd =  nu     * Az * cos(nu    .*t + psi);

figure
plot3(x*R/1000, y*R/1000, z*R/1000)
grid on
xlabel('x (km)')
ylabel('y (km)')
zlabel('z (km)')
axis equal
```

quintic.m

```
function x=quintic(chi)

% Lagrange's quintic equation, solved by fzero to find L1, L2, L3

global Ma
global Mb
global Mc

x=(Ma + Mb)*chi^5 + (3*Ma + 2*Mb)*chi^4 + (3*Ma + Mb)*chi^3 - (Mb + 3*Mc)*chi^2 - (2*Mb +
3*Mc)*chi - (Mb + Mc);
```

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