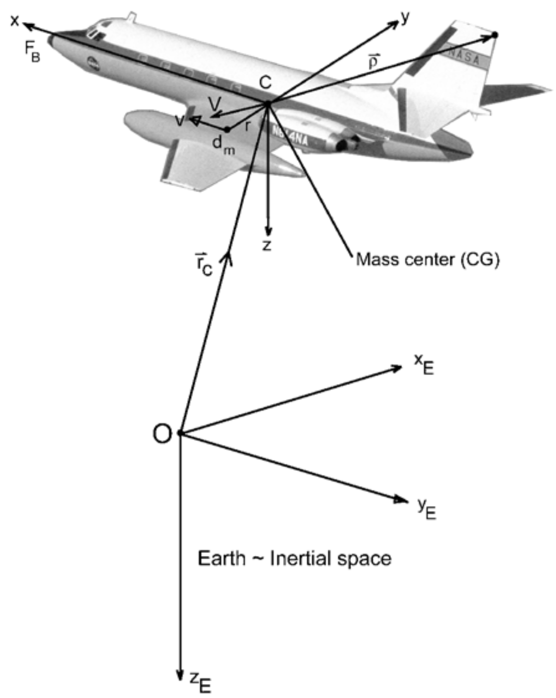
AEROSPACE DYNAMICS

EXAMPLE: GIVE ACCELERATION OF THE TIP OF THE RUDDER ON THIS AIRCRAFT



- LOOKING FOR ABSOLUTE ACCELERATION WITH RESPECT TO THE INERTIAL FRAME (EARTH IN THIS CASE)
- DEFINE A BUNCH OF POINTS AND VECTORS OF INTEREST
  - $\vec{r}_c$ - LOCATION OF A/C COM WRT THE ORIGIN
  - $\vec{\rho}$ - LOCATION OF RUDDER TIP WRT A/C COM

- ATTACH A FRAME TO THE AIRCRAFT (x, y, z)
  - CALLED A "BODY FRAME"
- ASSUME THAT AIRCRAFT MASS CENTER HAS VELOCITY $\vec{V}_{cM}$ AND ANGULAR VELOCITY OF THE VEHICLE WRT INERTIAL SPACE IS $\vec{\omega}$
- NOTE THAT $\vec{r}_r = \vec{r}_{cM} + \vec{\rho}$

WANT TO FIND $\ddot{\vec{r}}_r^{\,I}$.

$(\dot{\cdot})^I$ - MEANS DERIVATIVE WRT TIME AS SEEN IN INERTIAL FRAME

- TO COMPUTE $\ddot{\vec{r}}_r^{\,I}$ (INERTIAL ACCELERATION) WE WILL START BY FIRST COMPUTING $\dot{\vec{r}}_r^{\,I}$

- <u>KEY POINT</u>: OBSERVER ON THE AIRCRAFT CAN TELL US WHAT THE RUDDER IS DOING WITH RESPECT TO THE A/C CoM
  - BUT WE NEED TO ACCOUNT FOR THE AIRCRAFT MOTIONS AS WELL.

- KNOW THAT: $\dot{\vec{r}}_r^{\,I} = \dot{\vec{r}}_{cm}^{\,I} + \dot{\vec{\rho}}^{\,I}$

  QUESTION - HOW DO WE RELATE $\dot{\vec{\rho}}^{\,I}$ TO THE CHANGES THAT WE CAN SEE IN THE AIRCRAFT?

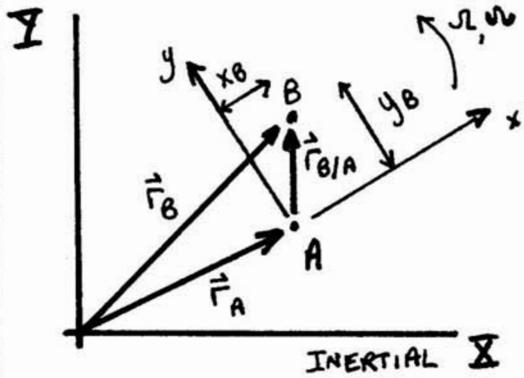
  - CHANGES AS SEEN BY AN OBSERVER ARE DENOTED AS $\dot{\vec{\rho}}^{\,B}$

- <u>KEY POINT</u>: MUST BE CAREFUL THAT THE DIFFERENTIATION IS CARRIED OUT IN THE PROPER REFERENCE FRAME.

- SOLUTION IS TO USE THE "TRANSPORT THEOREM" OR "CORIOLIS LAW"

$$\Rightarrow \quad \dot{\vec{\rho}}^{\,I} = \dot{\vec{\rho}}^{\,B} + \vec{\omega} \times \vec{\rho}$$

- WHERE DOES THIS EXPRESSION COME FROM?
  - CONSIDER SIMPLER 2D CASE
  - 2 POINTS "A", "B" WITH A FIXED AND B FREE TO ROTATE AROUND A



$$\Rightarrow \quad \vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

- AT THIS INSTANT, POINT A HAS ZERO VELOCITY AND ANGULAR VELOCITY / ACCELERATION ABOUT Z-AXIS OF $\vec{\Omega}$, $\dot{\vec{\Omega}}$

- FRAMES: X, Y INERTIAL, x, y AT "A" AND ROTATING.

- OBJECTIVE IS TO FIND

$$\vec{V}_B = \dot{\vec{r}}_B^{\,I} = \cancel{\dot{\vec{r}}_A^{\,I}}^{\,0} + \dot{\vec{r}}_{B/A}^{\,I} \equiv \frac{d^I}{dt}\left(\vec{r}_{B/A}\right)$$

- $\vec{r}_{B/A} = x_B \vec{i} + y_B \vec{j}$

  ⇒ "PROJECTION" OR REPRESENTATION OF VECTOR $\vec{r}_{B/A}$ IN TERMS OF THE CURRENT DIRECTIONS OF THE FRAME ATTACHED TO "A"

- TO COMPUTE THE DERIVATIVE $\frac{{}^{I}d}{dt}(\vec{r}_{B/A})$ WE MUST ACCOUNT FOR CHANGES IN BOTH
  - x_B, y_B
  - $\vec{i}, \vec{j}$

- $$\frac{{}^{I}d}{dt}(\vec{r}_{B/A}) = \underbrace{\frac{{}^{I}d x_B}{dt}\vec{i} + \frac{{}^{I}d y_B}{dt}\vec{j}}_{} + \underbrace{x_B \frac{{}^{I}d\vec{i}}{dt} + y_B \frac{{}^{I}d\vec{j}}{dt}}_{}$$

TIME RATE OF CHANGE WITH FRAME FIXED.
- SAME FOR BOTH OBSERVERS

⇒ SAME AS VELOCITY OF POINT B AS MEASURED BY AN OBSERVER IN FRAME A

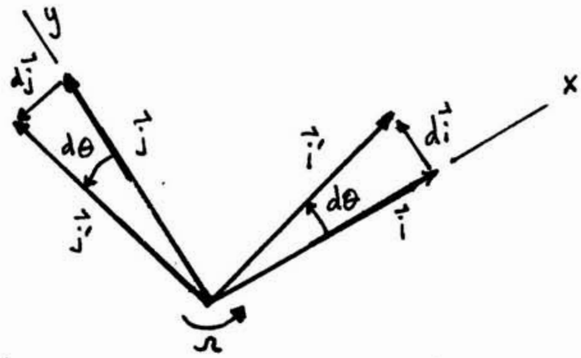
$\dot{\vec{r}}^{A}_{B/A}$

+

INSTANTANEOUS TIME RATE OF CHANGE OF UNIT VECTORS THAT DEFINE FRAME "A" AS MEASURED BY INERTIAL OBSERVER

- OK SO FAR, BUT HOW DO THESE UNIT VECTORS CHANGE WITH TIME?
  - FRAME A KNOWN TO BE ROTATING WITH RATE $\vec{\Omega} = \Omega \vec{K}$
  - DUE TO THE ROTATION

$$\vec{i} \rightarrow \vec{i}' = \vec{i} + d\vec{i}$$
$$\vec{j} \rightarrow \vec{j}' = \vec{j} + d\vec{j}$$



- NOTE: - $d\vec{i}$ AND $d\vec{j}$ ARE DUE ONLY TO THE INSTANTANEOUS ROTATION $d\theta$ ABOUT THE Z-AXIS.
  - ASSUMED TO BE A VERY SMALL ROTATION OVER A SHORT TIME PERIOD.

- CAN WRITE THAT $|d\vec{i}| = 1\,|d\theta|$ ARC LENGTH
- DIRECTION OF $d\vec{i}$ IS IN THE $+\vec{j}$ DIRECTION WHICH IS TANGENT TO PATH FOLLOWED BY TIP OF $\vec{i}$.

$$\Rightarrow \quad d\vec{i} = \vec{j}\,d\theta \qquad \text{SO} \qquad \frac{{}^{I}d\vec{i}}{dt} = \frac{{}^{I}d\theta}{dt}\vec{j}$$

WHAT ABOUT $d\vec{j}$?

WHAT IS $\frac{{}^{I}d\theta}{dt}$?

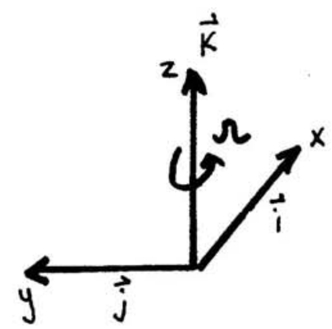
- NOTE $\frac{{}^{I}d\theta}{dt} = \Omega$ WHY?

AND $\frac{{}^{I}d\vec{i}}{dt} = \Omega\vec{j}$; $\frac{{}^{I}d\vec{j}}{dt} = -\Omega\vec{i}$

$\Rightarrow$ MORE COMPACT FORM IS :

$$\frac{{}^{I}d\vec{i}}{dt} = \vec{\Omega} \times \vec{i}$$

$$\frac{{}^{I}d\vec{j}}{dt} = \vec{\Omega} \times \vec{j}$$



- SO CAN WRITE

$$x_B \frac{{}^{I}d\vec{i}}{dt} + y_B \frac{{}^{I}d\vec{j}}{dt} = \vec{\Omega} \times x_B\vec{i} + \vec{\Omega} \times y_B\vec{j}$$
$$= \vec{\Omega} \times \left(x_B\vec{i} + y_B\vec{j} \right) = \vec{\Omega} \times \vec{r}_{B/A}$$

- **SUMMARY** :

$$\vec{v}_B = \dot{\vec{r}}_B^{\,I} = \dot{\vec{r}}_{B/A}^{\,A} + \vec{\Omega} \times \vec{r}_{B/A} = \dot{\vec{r}}_{B/A}^{\,I}$$

- SAME AS OUR TRANSPORT THEOREM WITH

$$\vec{r}_{B/A} \rightsquigarrow \vec{\rho}$$

- SO WE CAN WRITE

$$\dot{\vec{\rho}}^{I} = \dot{\vec{\rho}}^{B} + \vec{\omega} \times \vec{\rho}$$

$$\Rightarrow \dot{\vec{r}}_r^{I} = \dot{\vec{r}}_{CM}^{I} + \dot{\vec{\rho}}^{B} + \vec{\omega} \times \vec{\rho}$$

- TO COMPUTE THE ACCELERATION, MUST TAKE THE SECOND DERIVATIVE:

$$\ddot{\vec{r}}_r^{I} = \ddot{\vec{r}}_{CM}^{I} + \frac{d^I}{dt}\left(\dot{\vec{\rho}}^{B}\right) + \frac{d^I}{dt}\left(\vec{\omega} \times \vec{\rho}\right)$$

$\Rightarrow$ APPLY TRANSPORT THEOREM IN EACH CASE:

$$\frac{d^I}{dt}\left(\dot{\vec{\rho}}^{B}\right) = \ddot{\vec{\rho}}^{B} + \vec{\omega} \times \dot{\vec{\rho}}^{B}$$

$$\frac{d^I}{dt}\left(\vec{\omega} \times \vec{\rho}\right) = \dot{\vec{\omega}}^{I} \times \vec{\rho} + \vec{\omega} \times \frac{d^I}{dt}\vec{\rho} \longleftrightarrow \dot{\vec{\rho}}^{I}$$

$$= \dot{\vec{\omega}}^{I} \times \vec{\rho} + \vec{\omega} \times \left(\dot{\vec{\rho}}^{B} + \vec{\omega} \times \vec{\rho}\right)$$

$$\therefore \ddot{\vec{r}}_r^{I} = \ddot{\vec{r}}_{CM}^{I} + \ddot{\vec{\rho}}^{B} + 2\,\vec{\omega} \times \dot{\vec{\rho}}^{B} + \dot{\vec{\omega}}^{I} \times \vec{\rho} + \vec{\omega} \times \left(\vec{\omega} \times \vec{\rho}\right)$$

- PROBABLY HAVE SEEN THIS EXPRESSION BEFORE
  - FUNDAMENTAL FORM OF THE RELATIONSHIP BETWEEN ACCELERATIONS AS VIEWED IN FRAMES THAT ARE ROTATING ($\vec{\omega}$) AND ACCELERATING ($\ddot{\vec{r}}_{CM}^{I}$, $\dot{\vec{\omega}}^{I}$) WRT EACH OTHER.

- INDIVIDUAL COMPONENTS:

① $\ddot{\vec{r}}^{I}_{cm}$ - ACCELERATION OF B FRAME WRT I

② $\ddot{\vec{\rho}}^{B}$ - ACCELERATION OF RUDDER AS SEEN BY OBSERVER IN B AT C of M.

③ $2\vec{\omega} \times \dot{\vec{\rho}}^{B}$ - CORIOLIS ACCELERATION

④ $\dot{\vec{\omega}}^{I} \times \vec{\rho}$ - ANGULAR ACCELERATION

⑤ $\vec{\omega} \times (\vec{\omega} \times \vec{\rho})$ - CENTRIPETAL ACCELERATION.

- WILL SPEND TIME ANALYZING THESE IN DETAIL
  - ORDER OF MAGNITUDE?
  - CENTRIPETAL ACCELERATION AT EARTH'S SURFACE?

- KEY IS TO "CORRECTLY" SELECT THE FRAMES AND TO IDENTIFY ALL ANGLES/ANGULAR RATES BETWEEN THEM.

  ⇒ THERE ARE FORMAL WAYS (I.E. STANDARD WAYS) OF TRACKING AND DEFINING THESE ANGLES.

- FOCUS OF THIS CLASS WILL BE ON THE METHODOLOGY OF PROBLEM SOLVING.

  - F A R M

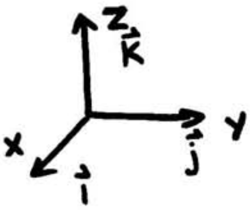
  - FRAMES
  - ANGLES
  - ROTATIONS
  - MECHANICS

⇒ TRICK IS TO BE SYSTEMATIC IN YOUR APPROACH TO THE PROBLEMS AND THAT WILL HELP YOU SOLVE THEM FASTER + BETTER.

① VECTOR - QUANTITY THAT HAS BOTH A DIRECTION AND A MAGNITUDE. $\vec{F}$

- INDEP. OF COORDINATE FRAME

② COORDINATE FRAME - RIGHT HANDED TRIAD OF UNIT VECTORS (ORTHOGONAL) THAT SPAN 3-SPACE



③ INERTIAL FRAME - ANY FRAME IN WHICH MOTION CAN BE DESCRIBED BY NEWTON'S LAWS

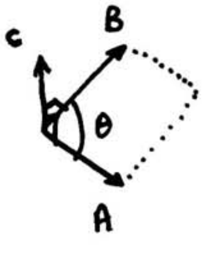
- IN GENERAL, FIXED RELATIVE TO THE STARS
- OFTEN FIND THAT FRAMES FIXED TO THE EARTH ARE "INERTIAL ENOUGH"

NOT ACCELERATING OR ROTATING

$\Rightarrow$ FUNCTION OF TIMESCALE + LENGTH OF MOTIONS UNDER CONSIDERATION.

④ $\vec{C} = \vec{A} \times \vec{B} \Rightarrow |\vec{C}| = |\vec{A}| \cdot |\vec{B}| \sin\theta$

θ - ANGLE BETWEEN $\vec{A}$ AND $\vec{B}$



$|\vec{C}|$ = AREA OF PARALLELOGRAM DEFINED BY $\vec{A}, \vec{B}$

- CROSS PRODUCT $\vec{C} = \vec{A} \times \vec{B}$

$$\vec{C} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \vec{i} \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} - \vec{j} \begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} + \vec{k} \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}$$

- CAN SHOW THAT IF REPRESENTATION OF VECTOR $\vec{A} = A_1 \vec{i} + A_2 \vec{j} + A_3 \vec{k}$

$\Rightarrow$ MATRIX FORM $\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$ GIVES THE COMPONENTS IN THE $\vec{i}, \vec{j}, \vec{k}$ FRAME

AND $\vec{B}$ HAS REPRESENTATION $\begin{bmatrix} B_1 \\ B_2 \\ B_2 \end{bmatrix}$

<u>THEN</u> MATRIX REPRESENTATION OF $\vec{C}$ GIVEN BY

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 & -A_3 & A_2 \\ A_3 & 0 & -A_1 \\ -A_2 & A_1 & 0 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

$$= A^{\times} B$$