Time-varying $\vec{B}$ field. Adiabatic invariants.

We have dealt so far with steady fields. When $\vec{B}$ varies in time, an additional effect comes into play, namely, the generation of electric fields according to $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$. If these variations are slower than the particle gyrations, orbit-averaging methods similar to those used for steady non-uniform fields can then be used, and the net energy added to the particles by the $\vec{E}$ field is small; the effects are called "adiabatic".

The second adiabatic invariant

We first consider the time dependent version of the steady conservation of the magnetic moment, that we analyzed in the previous lecture. A formally simple treatment consists of changing our reference frame to that of the parallel flow, $v_\parallel$, in which we see no flow, but a time variation of B. To see this, call z' the coordinate following $v_\parallel$, so that $z' = z - v_\parallel t$, and therefore $\left(\frac{\partial}{\partial t}\right)_{z'} = v_\parallel\left(\frac{\partial}{\partial z}\right)$. Since we had in the original frame $\frac{\partial \mu}{\partial z} = 0$, simple multiplication times $v_\parallel$ gives for the new, time-dependent situation, $\left(\frac{\partial \mu}{\partial t}\right)_{z'} = 0$, which is the time-dependent "adiabatic invariance". The magnetic moment $\mu = \frac{m w_\perp^2}{2B}$ is called "the second adiabatic invariant".

This formal treatment actually hides from view the real physics, so we next look at the situation in some more detail. As we noted above, the time variation of $\vec{B}$ produces an $\vec{E}$ field, and it is easy to see from $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$, this field circles around $\vec{B}$. As the particle executes its "quasi-Larmor orbit", its perpendicular kinetic energy $K_\perp$ increases at the rate $dK_\perp/dt = \vec{w} \cdot q\vec{E}$, and the increase in one orbit is,

$$\delta K_\perp = q \oint \vec{E} \cdot d\vec{l} = q \int\int \nabla \times \vec{E} \cdot d\vec{A} = -q \int\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

where $d\vec{l}$ is a path element along the Larmor circle, and $d\vec{A} = -dA\frac{\vec{B}}{B}$ is a surface element vector that points against the magnetic direction for a positive q. The integration can now be performed as,

$$\delta K_\perp = +q\frac{\partial B}{\partial t}\pi r_L^2 = \pi q \left(\frac{m w_\perp^2}{qB}\right)^2 \frac{\partial B}{\partial t} = 2\pi \frac{K_\perp}{\omega_c}\frac{1}{B}\frac{\partial B}{\partial t}$$

Now, $2\pi/\omega_c$ is the gyro time, and so the change in B over one gyration is $(2\pi/\omega_c)(\partial B/\partial t) = \delta B$. We thus have,

$$\frac{\delta K_\perp}{K_\perp} = \frac{\delta B}{B} \rightarrow \delta\left(\frac{K_\perp}{B}\right) = 0$$

which proves the invariance of μ.

The first adiabatic invariant

We have already seen that, in an inhomogeneous $\vec{B}$ field, averaging the Lorentz force $\vec{E}+\vec{w}\times\vec{B}$ over a Larmor orbit generates non-zero mean forces, which give wrist to the guiding center drifts. Once characteristic of these drifts is that they are generally slow enough that the inertia forces $m dv_D/dt$ are negligible, so that drifting motion is"quasi-static", like the ascent of a balloon or the fall of a raindrop, where to a good approximation, there is a force balance, even when the velocity may change with time. Thus, leaving aside the effects of strong enough Larmor-averaged forces to cause appreciable acceleration, we can write,

$$\vec{E} + \vec{v}_D \times \vec{B} \simeq 0$$

and we repeat here that in a non-steady situation, $\vec{E}$ is caused by the time-dependence of $\vec{B}$. Consider now a piece Σ of drifting material surface, meaning a surface that moves everywhere at the local drift velocity. Let $d\vec{A}$ be a surface element on Σ. The magnetic flux linked by Σ is then,

$$\Phi = \int\int \vec{B} \cdot d\vec{A}$$

and its rate of change is due to two effects: (a) the local rate of change of $\vec{B}$, and (b) the addition to new elements of surface to Σ along its edges as the piece of surface drifts; since the local velocity of the surface is $\vec{v}_D$, the new surface enclosed due to the motion of an element $d\vec{l}$, and so,

$$\frac{d\Phi}{dt} = \int\int_\Sigma \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} + \oint \vec{B} \cdot (\vec{v}_D \times d\vec{l})$$

We not replace $\partial\vec{B}/\partial t$ by $-\nabla \times \vec{E}$ in the first term and use Stokes' theorem. We also rotate the terms and flip the sign in the second integral, after which the two integrals can be combined into one:

$$\frac{d\Phi}{dt} = -\oint (\vec{E} + \vec{v}_D \times \vec{B}) \cdot d\vec{l} \simeq 0$$

The flux linked by a drifting material curve is therefore called the first adiabatic invariant.

A simple consequence of these two invariants is the possibility of heating and compressing a plasma by placing it in a rapidly increasing magnetic field (a so-called Z-pinch). The perpendicular temperature $T_\perp$ is proportional to the perpendicular mean kinetic energy, which, by the invariance of the magnetic moment, will increase with B. At the same time, from the first invariant, the plasma as a body will move with the magnetic lines of force, which will be squeezed together as B increases; hence the compression. In a purely cylindrical geometry, the plasma density will now increase in proportion to B, like $T_\perp$, and so, if we represent the density-temperature relationship as a polytropic formula ($\rho \sim T^{1/(\gamma-1)}$), we see that the appropriate "γ" is 2 for this compression process.

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