Effects of an Inhomogeneous Magnetic Field ($\vec{E} = 0$)

For some purposes the motion of the "guiding" centers can be taken as a good approximation of that of the particles. But it must be recognized that during the particle's Larmor gyrations, it "samples" regions where $\vec{B}$ or $\vec{E}$ may be different from what they are at the guiding center (GC). For the magnetic field, the effects are important, and they are to be studied here.

We decompose formally the particle's velocity $\vec{v}$ into a "Guiding Center" part $\vec{v}_{GC}$ and a Larmor part $\vec{v}_L$. In turn, the GC velocity has a component $\vec{v}_{||}$ along $\vec{B}$, plus a component (a "drift") perpendicular to $\vec{B}$:

$$\begin{aligned} \vec{v} &= \vec{v}_{GC} + \vec{v}_L \\ \vec{v}_{GC} &= \vec{v}_{||} + \vec{v}_D \end{aligned} \tag{1}$$

The Larmor velocity is defined by,

$$\begin{aligned} m\frac{d\vec{v}_L}{dt} &= q\vec{v}_L \times \vec{B}_{GC} \\ v_{L_{||}} &= 0 \end{aligned} \tag{2}$$

where $\vec{B}$ has been explicitly taken to be at the particle's Guiding Center.

We next assume the plasma is magnetized with respect to the particles in question, so that the Larmor radius is "small":

$$R_L << \text{Distance over which } \vec{B} \text{ changes appreciably} \tag{3}$$

We also assume (to be verified later) that the drift velocities are small compared to those involved in Larmor motion:

$$v_D << v_L \tag{4}$$

Because of (3), $\vec{B}(\vec{r})$ ($\vec{r}$ being the instantaneous location of the particle, measured from the GC) can be expanded to first order about $\vec{B}_{GC}$:

$$\vec{B}(\vec{r}) \cong \vec{B}_{GC} + (\vec{r}\cdot\nabla)\vec{B} \tag{5}$$

The full equation of motion is then,

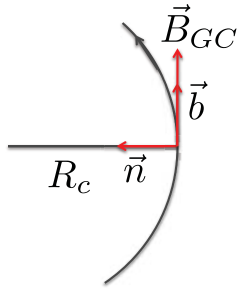
$$m\frac{d(\vec{v}_{||} + \vec{v}_D + \vec{v}_L)}{dt} = q(\vec{v}_{||} + \vec{v}_D + \vec{v}_L) \times (\vec{B}_{CG} + (\vec{r}\cdot\nabla))\vec{B} \tag{6}$$

Of the terms on the right, clearly $\vec{v}_{||} \times \vec{B}_{GC} = 0$ (parallel vectors). The term $\vec{v}_D \times (\vec{r}\cdot\nabla)\vec{B}$ is of <u>second order</u> (product of small terms), and will be neglected.

The term $q(\vec{v}_L \times \vec{B}_{GC})$ cancels the $md\vec{v}_L/dt$ on the left, according to (2).

This leaves,

$$\frac{d\vec{v}_{||}}{dt} + \frac{d\vec{v}_D}{dt} = \frac{q}{m}\left[\vec{v}_{||} \times (\vec{r}\cdot\nabla)\vec{B} + \vec{v}_D \times \vec{B}_{GC} + \vec{v}_L \times (\vec{r}\cdot\nabla)\vec{B}\right] \tag{7}$$



If the $\vec{B}$ line through the GC has a radius of curvature R_c, then,

$$\frac{d\vec{v}_{||}}{dt} \;=\; \frac{dv_{||}}{dt}\vec{b} \;+\; \frac{v_{||}^2}{R_c}\vec{n} \tag{8}$$

where $\vec{b}$ and $\vec{n}$ are unit vectors tangent to $\vec{B}$ and along the inner normal to the $\vec{B}$ line, respectively.

We can now separate out the components of Eq.(7) along $\vec{B}$ ($||$) and perpendicular to $\vec{B}$ ($\perp$)

Along $\vec{B}$:

$$\frac{dv_{||}}{dt} \;=\; \frac{q}{m}\left[\vec{v}_L \times (\vec{r}\cdot\nabla)\vec{B}\right]_{||} \tag{9}$$

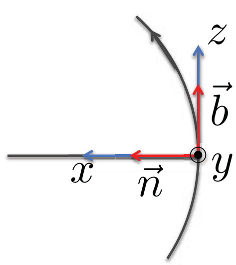
Perpendicular to $\vec{B}$:

$$\frac{v_{||}^2}{R_c}\vec{n} \;+\; \frac{d\vec{v}_D}{dt} \;=\; \frac{q}{m}\left[\vec{v}_{||}\times(\vec{r}\cdot\nabla)\vec{B} \;+\; \vec{v}_D\times\vec{B}_{GC} \;+\; [\vec{v}_L\times(\vec{r}\cdot\nabla\vec{B})]_{\perp}\right] \tag{10}$$

The next step is to average these equations over each "quasi-orbit" about the GC ("quasi", because, in an inhomogeneous field the Larmor orbits may not quite close on themselves - we ignore that). During each such orbit, $\vec{v}_{||}$, $\vec{v}_D$ and $\vec{B}_{GC}$ are regarded as constant, but $\vec{r}$ and $\vec{v}_L$ vary cyclically. Because of this, any term where $\vec{r}$ and $\vec{v}_L$ appear <u>linearly</u> will average to zero, but care must be taken when they appear in pairs. In particular,

$$\left\langle \vec{v}_{||} \;\times\; (\vec{r}\cdot\nabla)\vec{B}\right\rangle_{Larmor} \;=\; 0 \tag{11}$$

<u>Note:</u> If we had retained $\vec{E}$ in the formulation (even with some inhomogeneity), this averaging step would have eliminated any first order $\nabla\vec{E}$ (no $\nabla\vec{E}$ drift terms exist).



To see what the averaging results are for the other terms, we now re-write (9) and (10) in a local Cartesian coordinate frame as shown in the figure. The y direction completes a right-handed set. The corresponding unit vector is the <u>bi-normal</u> unit vector

$$\vec{bn} \;=\; \vec{b} \;\times\; \vec{n} \tag{12}$$

Writing out Eq. (9):

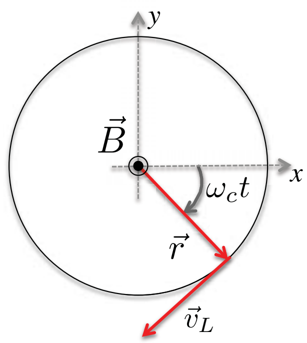
$$\frac{dv_z}{dt} \;=\; \frac{q}{m}\left\langle v_{L_x}\left(x\frac{\partial B_y}{\partial x} + y\frac{\partial B_y}{\partial y}\right) - v_{L_y}\left(x\frac{\partial B_x}{\partial x} + y\frac{\partial B_x}{\partial y}\right)\right\rangle_{Larmor} \tag{13}$$

Projecting Eq. (10) along the normal (x) direction,

$$\frac{v_z^2}{R_c} \;+\; \frac{dv_{D_x}}{dt} \;=\; \frac{q}{m}\left\{v_{D_y}B_z + \left\langle v_{L_y}\left(x\frac{\partial B_z}{\partial x} + y\frac{\partial B_z}{\partial y}\right)\right\rangle_{Larmor}\right\} \tag{14}$$

Projecting Eq. (10) along the bi-normal (y),

$$\frac{dv_{D_y}}{dt} = \frac{q}{m}\left\{ -v_{D_x}B_z - \left\langle v_{L_x}\left(x\frac{\partial B_z}{\partial x} + y\frac{\partial B_z}{dy}\right)\right\rangle_{Larmor}\right\} \tag{15}$$



The Larmor motion is shown in the figure for a positive q. If the Larmor radius is r_L, we have

$$\begin{aligned} x &= r_L\cos\omega_c t \\ y &= -r_L\sin\omega_c t \end{aligned} \tag{16}$$

$$\begin{aligned} v_{Lx} &= -\omega_c r_L \sin\omega_c t \\ v_{Ly} &= -\omega_c r_L \cos\omega_c t \end{aligned} \tag{17}$$

where $\omega_c = \frac{qB}{m}$ is the gyro frequency. Therefore, when doing the averaging,

$$\begin{aligned} \left\langle x v_{L_y}\right\rangle_L &= -\omega_c r_L^2 \left\langle \cos^2\omega_c t\right\rangle_L = -\frac{1}{2}\omega_c r_L^2 \\ \left\langle y v_{L_y}\right\rangle_L &= \omega_c r_L^2 \left\langle \sin\omega_c t \cos\omega_c t\right\rangle_L = 0 \\ \left\langle x v_{L_x}\right\rangle_L &= -\omega_c r_L^2 \left\langle \sin\omega_c t \cos\omega_c t\right\rangle_L = 0 \\ \left\langle y v_{L_x}\right\rangle_L &= \omega_c r_L^2 \left\langle \sin^2\omega_c t\right\rangle_L = \frac{1}{2}\omega_c r_L^2 \end{aligned} \tag{18}$$

Notice the occurrence of the group $\frac{q}{m}\frac{1}{2}\omega_c r_L^2$. From $\frac{q}{m} = \frac{\omega_c}{B}$ and $\omega_c r_L = v_\perp$, this group is,

$$\frac{q}{m}\frac{1}{2}\omega_c r_L^2 = \frac{\frac{1}{2}v_\perp^2}{B} = \frac{\mu}{m} \tag{19}$$

where μ is the magnetic moment $\mu = \frac{1}{2}mv_\perp^2/B$. We now have from (13), (14), and (15),

$$\begin{aligned} \frac{dv_{D_x}}{dt} &= -\frac{v_z^2}{R_c} + \omega_c v_{D_y} - \frac{v_\perp^2}{2B}\frac{\partial B_z}{\partial x} \quad (a) \\ \frac{dv_{D_y}}{dt} &= -\omega_c v_{D_x} - \frac{v_\perp^2}{2B}\frac{\partial B_z}{\partial y} \quad (b) \\ \frac{dv_{D_z}}{dt} &= \frac{v_\perp^2}{2B}\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right) = -\frac{v_\perp^2}{2B}\frac{\partial B_z}{\partial z} \quad (c) \end{aligned} \tag{20}$$

Notice we have used $\nabla\cdot\vec{B} = 0$ in the last equation. Also, since $B^2 = B_x^2 + B_y^2 + B_z^2$ and $B_x = B_y = 0$ due to the choice of axes, $B = B_z$. Furthermore, taking gradients,

$$2B\nabla B = 2B_x\nabla B_x + 2B_y\nabla B_y + 2B_z\nabla B_z = 2B\nabla B_z$$

and so $\nabla B_z = \nabla B$. Hence the subscript z will be omitted from B_z from here on.

These averaged equations contain rich information. Let us first consider separately (20a) and (20b), i.e. the $\perp$ part. Following the procedure used before, we define a <u>complex v_D vector</u>,

$$v_D \;=\; v_{D_x} + i v_{D_y} \tag{21}$$

and combine the equations as $(a) \; + \; i(b)$:

$$\begin{aligned} \frac{d}{dt}\Big(v_{D_x} + v_{D_y}\Big) \;&=\; -\frac{v_\|^2}{R_c} + \omega_c\Big(v_{D_y} - i v_{D_x}\Big) - \frac{v_\perp^2}{2B}\nabla_\perp B \\ \frac{dv_D}{dt} + i\omega_c v_D \;&=\; -\frac{v_\|^2}{R_c} - \frac{v_\perp^2}{2B}\nabla_\perp B \end{aligned} \tag{22}$$

where $\nabla_\perp \;=\; \frac{\partial}{\partial x}\vec{i} \,+\, \frac{\partial}{\partial y}\vec{j}$. The homogeneous solution of (22) is the Larmor motion, which we disregard once again, since we are after the drifts. These drifts are given by the <u>particular</u> solution,

$$\begin{aligned} v_D \;=\; v_{D,part} \;&=\; i\frac{v_\|^2}{\omega_c R_c} + i\frac{v_\perp^2}{2B\omega_c}\nabla_\perp B \\ v_D \;&=\; i\frac{m v_\|^2}{qBR_c} + i\frac{m v_\perp^2}{2qB^2}\nabla_\perp B \end{aligned} \tag{23}$$

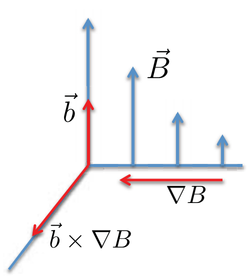
In vector terms,

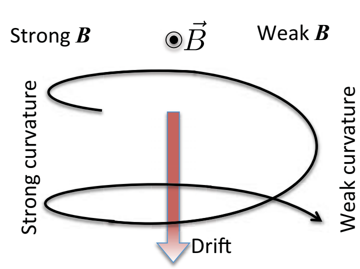
$$\vec{v}_D \;=\; \frac{m v_\|^2}{qBR_c}\vec{bn} + \frac{m v_\perp^2}{2qB^2}\vec{b}\times\nabla_\perp B \tag{24}$$

Of these, the first term is the <u>Inertia Drift</u> due to the centrifugal force on a particle which is sliding along a curved $\vec{B}$ line. One mechanistic way to view this is the following: The centrifugal force is $-\frac{m v_\|^2}{R_c}\vec{n}$. This can be countered by the magnetic force due to a drift along $\vec{bn}$, chosen such that,

$$\frac{m v_\|^2}{R_c}\vec{n} \;=\; q\vec{v}_{D_I}\times\vec{B} \;=\; q v_{D_I}(\vec{bn})\times B\vec{b}$$

and, from (12), $(\vec{bn})\times\vec{b} = \vec{n}$, leaving $v_{D_I} \;=\; \frac{m v_\|^2}{qBR_c}$, as in (24).

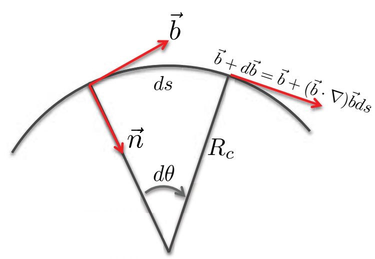




The second term in (24) is the so-called <u>∇B - Drift</u> (Notice that $\vec{b}\times\nabla_\perp B \;=\; \vec{b}\times\nabla B$). This will be later shown to be also related to $\vec{B}$ - line curvature, since, due to $\nabla\cdot\vec{B} = 0$, the magnitude of B can only vary if the lines are curved. This ∇B - drift is seen to be perpendicular to $\vec{b}$ and the ∇B and directed to the left of $\vec{B}$. A physical picture is in shown in the figure.

To show how ∇B is related to line curvature, we start from Frenet's first formula (illustrated in the figure):

$$\frac{\vec{n}}{R_c} = (\vec{b}\cdot\nabla)\vec{b} \tag{25}$$



and use the vector identity $\vec{B}\times(\nabla\times\vec{B}) = \nabla\left(\frac{B^2}{2}\right) - (\vec{B}\cdot\nabla)\vec{B}$. Dividing this by B, and remembering $\frac{\vec{B}}{B}=\vec{b}$, $\nabla\left(\frac{B^2}{2}\right) = B\nabla B$,

$$(\vec{b}\cdot\nabla)\vec{B} = \nabla B - \vec{b}\times(\nabla\times\vec{B})$$

and so,

$$(\vec{b}\cdot\nabla)\vec{b} = (\vec{b}\cdot\nabla)\left(\frac{\vec{B}}{B}\right) = \frac{1}{B}[\nabla B - \vec{b}\times(\nabla\times\vec{B})] - \frac{\vec{B}}{B^2}\vec{b}\cdot\nabla B$$

which is also $\frac{\vec{n}}{R_c}$, according to (25). We now cross-multiply times $\vec{b}$, to from $\frac{\vec{b}\times\vec{n}}{R_c} = \frac{(\vec{bn})}{R_c}$:

$$\frac{\vec{bn}}{R_c} = \frac{\vec{b}\times\nabla B}{B} - \frac{1}{B}\underbrace{\vec{b}\times\left(\vec{b}\times(\nabla\times B)\right)}$$

$$\vec{b}\,\vec{b}\cdot(\nabla\times B) - \nabla\times\vec{B} = -(\nabla\times\vec{B})_\perp$$

or

$$\frac{\vec{b}\times\nabla B}{B} = \frac{\vec{bn}}{R_c} - \frac{(\nabla\times\vec{B})_\perp}{B}$$

and substituting this into the expression for $(\vec{v}_D)_{\nabla B}$ (from (24)),

$$(\vec{v}_D)_{\nabla B} = \frac{mv_\perp^2}{2qB^2}\vec{b}\times\nabla B = \frac{mv_\perp^2}{2qB}\left(\frac{\vec{bn}}{R_c} - \frac{(\nabla\times\vec{B})_\perp}{B}\right) \tag{26}$$

In the absence of appreciable net current $\perp$ to $\vec{B}$, one would have $\left(\nabla\times\vec{B}\right)_\perp = 0$, and $(\vec{v}_D)_{\nabla B}$ would be strictly proportional to $\vec{B}$-line curvature ($1/R_c$), and directed along the binormal vector just as the inertia drift. We should therefore group these terms together. Starting at Eq. (24),

$$\vec{v}_D = \frac{m\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)}{qBR_c}\vec{bn} - \frac{mv_\perp^2}{2qB^2}(\nabla\times\vec{B})_\perp \tag{27}$$

<u>Parallel Drifts</u>

To complete the picture, we now return to Eq. (20c) and investigate the <u>parallel</u> drift effects. We had,

$$m\frac{dv_\parallel}{dt} = -\frac{mv_\perp^2}{2B}\frac{\partial B}{\partial z} \tag{28}$$

In steady state, $m\frac{dv_{||}}{dt} = mv_{||}\frac{dv_{||}}{dz} = \frac{d\left(\frac{1}{2}mv_{||}^2\right)}{dz}$. Since the total kinetic energy is conserved (no $\vec{E}$ forces, and $\vec{B}$ forces perpendicular to $\vec{v}$),

$$\frac{d\left(\frac{1}{2}mv_{||}^2\right)}{dz} = -\frac{d\left(\frac{1}{2}mv_{\perp}^2\right)}{dz}$$

Substituting in (28),

$$\frac{d\left(\frac{1}{2}mv_{\perp}^2\right)}{dz} = \left(\frac{1}{2}mv_{\perp}^2\right)\frac{1}{B}\frac{dB}{dz}$$

or

$$\frac{d}{dz}\left(\frac{\frac{1}{2}mv_{\perp}^2}{B}\right) = 0 \tag{29}$$

The quantity $\mu = \frac{\frac{1}{2}mv_{\perp}^2}{B}$ (the magnetic moment, μ, according to our previous definition) is seen to be invariant for a particle which moves along a $\vec{B}$-line while executing Larmor rotations perpendicular to $\vec{B}$. This is sometimes called the second adiabatic invariant ("adiabatic" refers to the absence of energy input, as from a time-varying $\vec{B}$ field).

There are now two constants of the motion:

$$E = \frac{1}{2}mv_{||}^2 + \frac{1}{2}mv_{\perp}^2 \tag{30}$$

and

$$\mu = \frac{\frac{1}{2}mv_{\perp}^2}{B} \tag{31}$$

Eliminating the perpendicular energy $\frac{1}{2}mv_{\perp}^2 = \mu B$, we obtain

$$\frac{1}{2}mv_{||}^2 = E - \mu B \tag{32}$$

Showing that the parallel energy decreases as the particle moves into a region with a stronger B field. At the same time, of course, the perpendicular energy increases: the varying field has the effect of transferring energy from ($\perp$) to ($||$) or viceversa.

If B increases enough, a particle of given energy and magnetic moment can be stopped in its parallel motion, and reflected. Suppose the particle has a parallel velocity $v_{||0}$ when the field is B_0. It will be reflected at turning point T if $\frac{1}{2}mv_{||0}^2 + \mu B_0 = \mu B_T$:

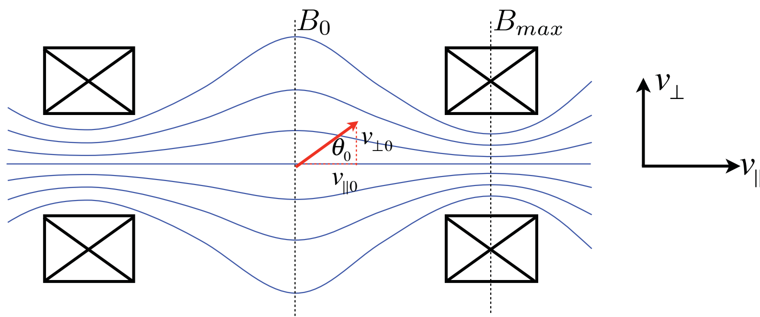
$$B_T = B_0 + \frac{\frac{1}{2}mv_{||0}^2}{\mu} = B_0 + \frac{v_{||0}^2}{v_{\perp 0}^2}B_0$$

or

$$\frac{B_T}{B_0} = 1 + \left(\frac{v_{||}}{v_{\perp}}\right)_0^2 \tag{33}$$

Defining the "pitch angle" $\theta = \text{atan}\left(v_\perp / v_{||}\right)$,

$$\frac{B_T}{B_0} = 1 + \cot\theta_0^2 = \frac{1}{\sin^2\theta_0} \tag{34}$$



Any particle for which $\sin\theta_0 > \sqrt{B_0/B_{max}}$ will be reflected at some B_T given by (34). But those with small enough pitch angle ($\sin\theta_0 < \sqrt{B_0/B_{max}}$) will not, and will escape through the magnetic bottleneck. If a "Magnetic Bottle" is used to confine plasma between solenoids, this "leakage"of low-pitch particles creates a peculiar distribution, with particles within a "loss-cone" in pitch being mostly absent.